BINGHAMTON UNIVERSITY

State University of New York

EECE 301
Signals & Systems

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Note Set #22

- D-T Systems: DT Systems & Difference Equations

D-T System Models

So far we've seen how to use the FT to analyze circuits…

- Use phasors and standard circuit analysis methods
- Find the Frequency Response of the circuit

But for D-T systems we can't use *circuit* analysis methods!

So… instead we use the fact that LTI DT systems are described by Difference Equations.

- Later we'll use that to learn to use DTFT to analyze LTI DT systems

A general N^{th} order Difference Equations looks like this:

$$\boxed{y[n]+a_1 y[n-1]+\ldots+a_N y[n-N]=b_0 x[n]+b_1 x[n-1]+\ldots+b_M x[n-M]}$$

Most "Advanced" Output Sample

Least "Advanced" Output Sample

The difference between these two index values is the "order" of the difference eq.
Here we have: $n - (n - N) = N$

<u>Exploring How Difference Equations "Work"</u>

Before we learn how to apply the DTFT to DT systems we first get a little more understanding of just how a difference equation works

Here we'll look at a numerical way to solve Difference Equations that is called "Recursion"… and it is actually used to implement (or build) many D-T systems, which is the main advantage of the recursive method.

The disadvantage of *<u>solving</u>* using the recursive method is that it doesn't provide a so-called "closed-form" solution… in other words, you don't get an equation that describes the output… you just get a finite-duration sequence of numbers that shows part of the output.

Later we'll see how to get "closed-form" solutions… such solutions give engineers keen insight needed to perform design and analysis tasks.

<u>Solution by Recursion</u>

But, for computer processing it is possible to <u>recursively</u> solve (i.e. compute) a <u>numerical</u> solution. In fact, this is how D-T systems are <u>implemented</u> (i.e. built!)

We can re-write any linear, constant-coefficient difference equation in "recursive form". Here is the form we've already seen for an N^{th} order difference:

$$y[n]+a_1 y[n-1]+\ldots+a_N y[n-N]=b_0 x[n]+b_1 x[n-1]+\ldots+b_M x[n-M]$$

Re-Write As:

$$y[n]+\sum_{i=1}^{N} a_i y[n-i]=\sum_{i=0}^{M} b_i x[n-i]$$

Now… isolating the $y[n]$ term gives the "Recursive Form":

$$\boxed{y[n]=\underbrace{-\sum_{i=1}^{N} a_i y[n-i]}_{\text{Some "past" output values, with values already known}}+\underbrace{\sum_{i=0}^{M} b_i x[n-i]}_{\text{current \& past input values already "received"}}}$$

"<u>current</u>" <u>output</u> value to be computed

Some "<u>past</u>" <u>output values</u>, with values already known

current & past input values already "received"

The key to Recursive Form is that you have the current output $y[n]$ in terms of <u>*past*</u> outputs $y[n - i]$

Note: sometimes it is necessary to re-index a difference equation using $n+k \rightarrow n$ to get this form… as shown below.

Here is a slightly different form… but it is still a difference equation:

$$y[n+2]-1.5y[n+1]+y[n]=2x[n]$$

If you isolate $y[n]$ here you will get the current output value in terms of <u>future</u> output values (Try It!)… We don't want that!

So… in general we start with the "Most Advanced" output sample… here it is $y[n+2]$… and re-index it to get only n (of course we also have to re-index everything else in the equation to maintain an equation):

So here we need to subtract 2 from each sample argument:

$$y[n]-1.5y[n-1]+y[n-2]=2x[n-2]$$

Now we can put this into recursive form as before.

Ex: Solve this difference equation recursively

$$y[n]-1.5y[n-1]+y[n-2]=2x[n-2]$$

For $x[n]=u[n]$ unit step

And ICs of: $\begin{cases} y[-2]=2 \\ y[-1]=1 \end{cases}$

Note: You need N "past" values as IC's to solve an N^{th} order Difference Equation

Recursive Form: $y[n]=1.5y[n-1]-y[n-2]+2x[n-2]$

n	$x[n]=u[n]$	$y[n]$
-2	0	2
-1	0	1
0	1	$1.5\cdot 1-2+2\cdot 0=-0.5$
1	1	$1.5\cdot(-0.5)-1+2\cdot 0=-1.75$
2	1	-.0125
3	1	3.563

1st: Fill in Input (Unit Step here)

2nd: Put IC's Here

3rd: Compute n=0 Output
$y[0]=1.5y[-1]-y[-2]+2x[-2]$

4th: Compute n=1 Output
$y[1]=1.5y[0]-y[-1]+2x[-1]$

Etc.

We can write a simple matlab routine to implement this difference equation $y[n] = 1.5y[n-1] - y[n-2] + 2x[n-2]$

There is a more general version of this code on the Book's web page.

```
function y = recur_2(x,y_ics);

y(1) = y_ics(1);
y(2) = y_ics(2);

for k=3:(length(x)+2)
        y(k)=1.5*y(k-1)-y(k-2)+2*x(k-2);
end
```

x is a vector of input samples
(from our table-based solution we see that we need the vector x to start at n = -2)
y_ics is 1x2 vector holding the 2 ICs

y will be the returned vector holding the output samples

Write the ICs into the output vector's first two positions

Each time through the for-loop we compute the output value according to the recursive form of the difference equation

```
x = [0 0 ones(1,20)];
```

```
stem(-2:(length(y)-3),y)
```

The trickiest part of getting this code right is getting the indexing right!!!

Mathematical indexing used in difference equations is “zero-origin” and allows negative indices.

Matlab indexing is “one-origin” and does NOT allow negative indexing.

The “k” in the code is related to the math index *n* according to: k = n+3

Thus, when we first enter the loop we are computing for k=3 or *n* = 0

```
function y = recur_2(x,y_ics);

y(1) = y_ics(1);
y(2) = y_ics(2);

for k=3:(length(x)+2)
        y(k)=1.5*y(k-1)-y(k-2)+2*x(k-2)
end
```

Store *y*[-2] in k=1 position of vector
Store *y*[-1] in k=2 position of vector

We must continue the loop until the last input value is used… since we use x(k-2) in the recursion we need to stop our for-loop at length(x)+2.

We already have filled the first two elements of the output vector so we put y[0] into the 3^{rd} position, etc.

That way… when we go through the last loop (i.e., k = length(x)+2) we’ll index x using k-2 = length(x)… which grabs the <u>last</u> element in the input vector x

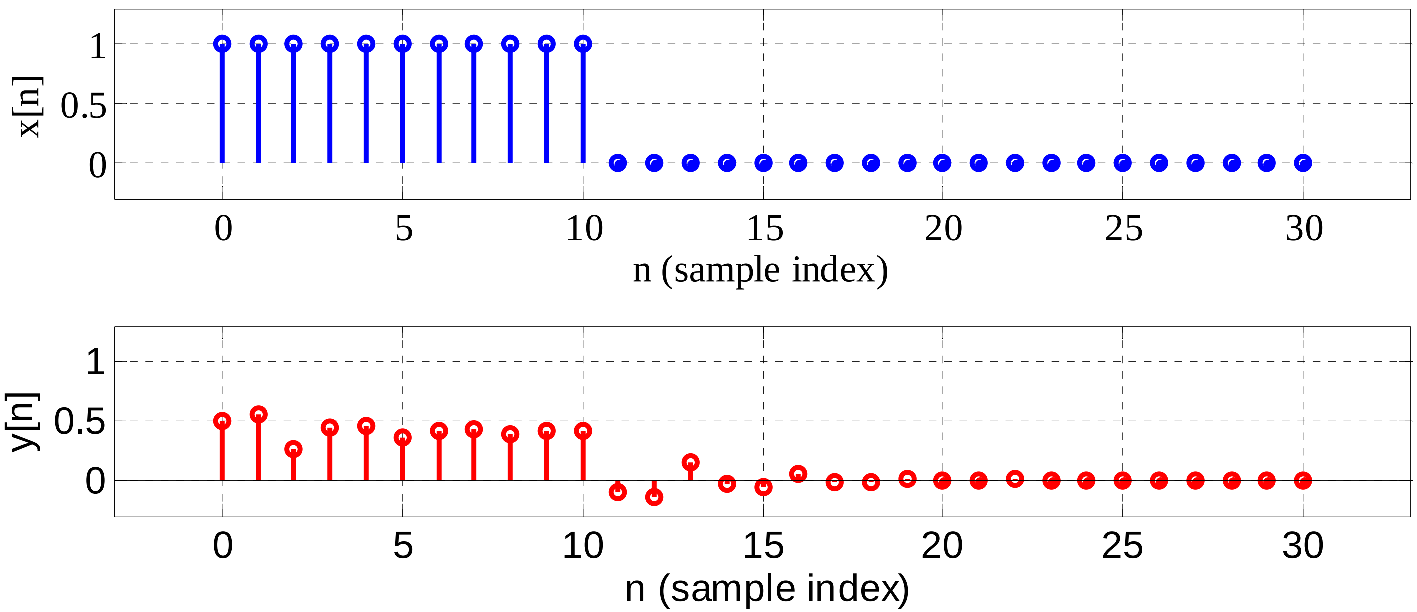
MATLAB "filter" Command

y=filter(b,a,x)

MATLAB has a function to implement an LTI DT system defined by a Diff Eq.

$$y[n]+0.7y[n-1]+0.5y[n-2]=0.5x[n]+0.4x[n-1] \qquad x[n]=\begin{cases}1, & n=0,1,2,\ldots,10\\ 0, & elsewhere\end{cases}$$

```
>> x = [ones(1,11)  zeros(1,20)];
>> y = filter([0.5 0.4],[1 0.7 0.5],x);
>> subplot(2,1,1); stem(0:30,x,'b')
>> subplot(2,1,2); stem(0:30,y,'r')
```



We could use these ideas to implement this D-T system on a computer… although for real-time operation we would not use matlab, we likely would write the code using C or assembly language.

Also… we probably wouldn't implement this on a general microprocessor like those used in desktop or laptop computers. We would implement it in a microcontroller for simple applications but for high-performance signal processing applications (like for radar and sonar, etc.) we would use a special DSP microprocessor.

[Web Link to Extra Info on DSP Processors]

This is a S/W implementation of the D-T system…. It is also possible to build dedicated digital H/W to implement it.

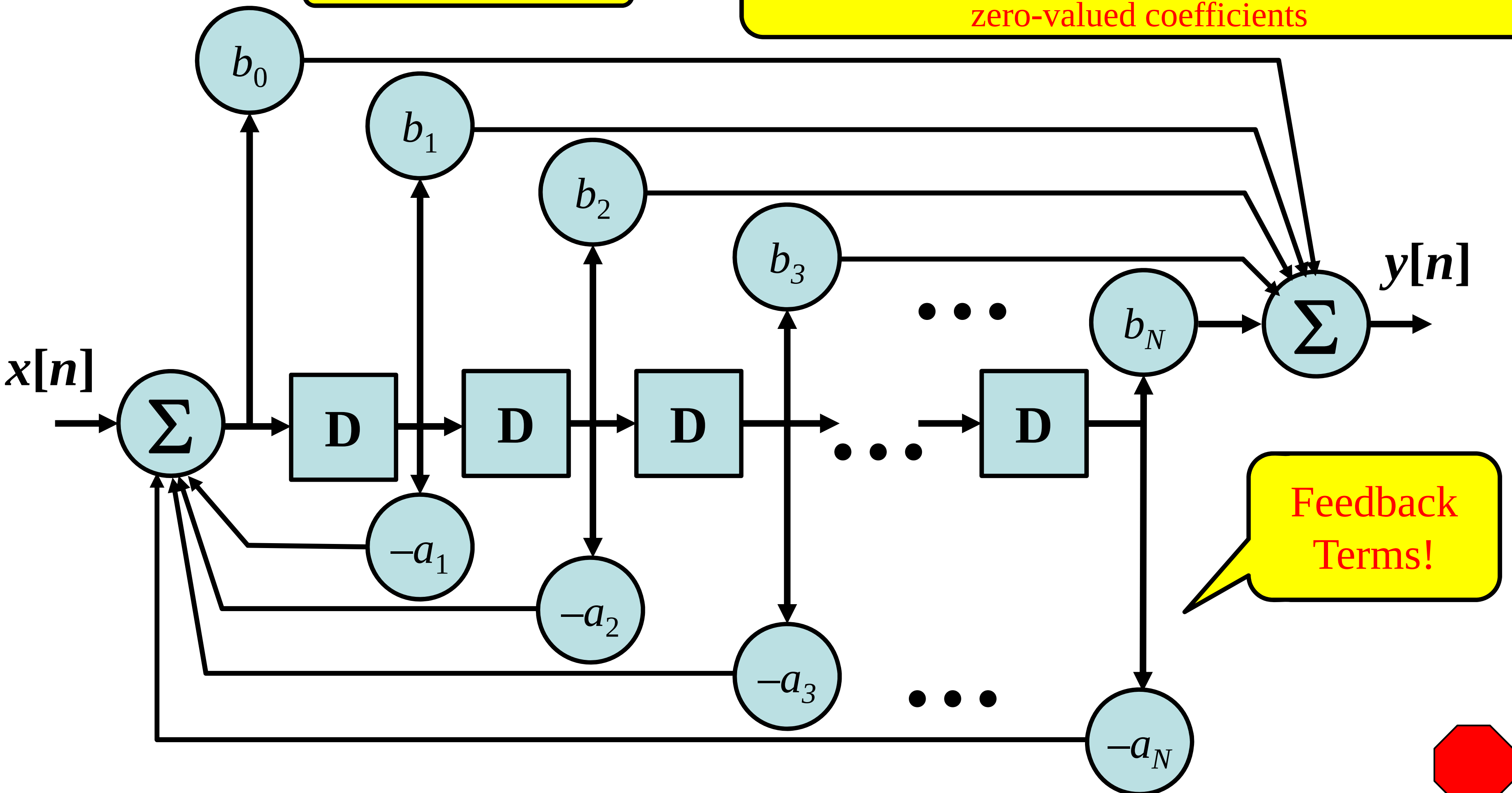
[Web Link to Example of Dedicated H/W D-T System]

Block Diagram ("Hardware") View of DT System

$$y[n]+a_1y[n-1]+\ldots+a_Ny[n-N]=b_0x[n]+b_1x[n-1]+\ldots+b_Nx[n-N]$$

Feedback Terms!

We show the same # of terms on each side… can always force this case by extending one side with zero-valued coefficients



D = Delay ("Memory Latch")

All Elements are "Clocked" (not shown)