Cramer-Rao Lower Bounds for Estimation of Doppler Frequency in Emitter Location Systems

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Abstract: This paper derives Cramer-Rao bounds on estimates of the Doppler-shifted frequency of a coherent pulse-train intercepted at a single moving antenna. Such estimates are used to locate the emitter that transmitted the pulse train. Coherency from pulse to pulse allows much better frequency accuracy and is considered to be necessary to support accurate emitter location. Although algorithms for estimating the Doppler-shifted frequency of a coherent pulse train have been proposed, previously no results were available for the Cramer-Rao lower bound (CRLB) for frequency estimation from a coherent pulse train. This paper derives the CRLB for estimating the Doppler-shifted frequency of a coherent pulse train as well as for a non-coherent pulse train; a comparison of these two cases is made and the bound is compared to previously published accuracy results. It is shown that a general rule of thumb is that the frequency CRLB for coherent pulse trains depends inversely on pulse on-time, number of pulses, variance of pulse times, and the product of signal-to-noise-ratio and sampling frequency $\text{SNR} \times F_s$; pulse shape and modulation have virtually no impact on the frequency accuracy. For the case that the $K$ intercepted pulses are equally spaced by the pulse repetition interval (PRI), then the CRLB decreases as $1/PRI^2$ and as $1/K^3$. It is also shown that roughly the gain in coherent accuracy vs. the non-coherent case is $K$ times the ratio of pulse on-time to PRI; since PRI is typically much larger than pulse on-time, the coherent scenario allows much better frequency estimation accuracy.

Index Terms: Emitter Location, Frequency Estimation, Doppler Shift, Cramer-Rao Bound

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I. Introduction

Passive location of a stationary emitter with unknown frequency from a single moving platform is an important problem that has been investigated recently [1] – [10]. The importance of single-platform methods stems from the desire to provide accurate location of emitters from a single aircraft for tactical and/or strategic uses: single aircraft ability provides flexibility over multi-platform methods [9] and therefore there has been much interest in the development and testing of single-platform methods [4], [5]. All the single-platform approaches that have been proposed consist of measuring, at instants within some interval of time, one or more signal features that depend on the emitter’s location according to some signal model (i.e., measurement model), and then processing them to estimate the emitter location. For example, the measured signal feature could be the bearing to the emitter [3], [9], the Doppler-shifted frequency of the signal [2], [6], [8], the interferometer phase between two spaced antennas on a single platform [2], the times-of-arrival of the pulses, etc., or combinations of these [1], [7].

Regardless of the feature(s) exploited, the general approach is the same: first estimate the signal features (e.g., frequency, angle, etc.) at a set of time instants and then use a signal model that describes the relationship between the measured signal feature(s) and the emitter location, together with some statistical inference technique (e.g., least-squares, maximum likelihood, etc.) to estimate the emitter’s location. Thus, to evaluate the location accuracy of an emitter location method it is first necessary to evaluate the accuracy of the feature(s) estimation.

This paper focuses on the specific case of measuring the Doppler-shifted frequency of a coherent pulse-train signal intercepted at a single moving antenna for use in emitter location. Algorithms for estimating the Doppler-shifted frequency of a coherent pulse train have been proposed [6], [11], where coherency from pulse to pulse allows much better frequency accuracy and is considered to be necessary to support the subsequent emitter location processing. Despite the publication of these algorithms for coherent frequency estimation, no results have been available for the Cramer-Rao lower bound (CRLB) for this case. The only accuracy results
previously available were: simulation results from Becker’s algorithm [11], experimentally obtained accuracy levels reported by corporations developing actual systems [4], [5], and an approximate accuracy analysis [10]. This paper derives the CRLB for estimating the Doppler-shifted frequency of a coherent pulse train and compares the bound to previously published accuracy results; for comparison, the CRLB is also derived for a non-coherent pulse train and a simple formula is derived that characterizes the improvement achievable due to exploiting coherency.

II. CRLB for Frequency Estimation of Coherent Pulse Train

A. Signal Model

A complex-valued coherent pulse train from an emitter, consisting of \( K \) pulses, can be modeled by

\[
s(t) = e^{j(\omega t + \phi)} \sum_{k=0}^{K-1} p(t - T_k),
\]

where \( p(t) \) is a single pulse (possibly complex), \( \omega \) is the carrier frequency, \( \phi \) is a phase offset, and the \( T_k \) are called the transmitted pulse times. Let \( T_{OT} \) be the total pulse “on-time” and assume that \( p(t) \) is time-limited on the interval \([0, T_{OT}]\) with \( T_{OT} < T_k - T_{k-1} \). The signal is received at a moving platform, with initial range \( d_0 \), and radial velocity \( v_0 \). The (noise-free) received signal is model as

\[
r(t) = s(\alpha t - \frac{d_0}{c}),
\]

where \( \alpha = 1 - v_0 / c \) is the signal dilation coefficient from the Doppler effect, and \( c \) is the signal propagation rate. Substituting (1) into (2) yields that at the receiver the signal appears as

\[
r(t) = e^{j\alpha t} e^{j\phi} \sum_{k=0}^{K-1} \tilde{p}(t - \hat{T_k}),
\]
where \( \tilde{\omega} = \alpha \omega \), \( \tilde{\phi} = \phi - \omega d_0 / c \), \( \tilde{p}(t) = p(\alpha t) \) and \( \tilde{T}_k = \left( d_0 / c + T_k \right) / \alpha \); the \( \tilde{T}_k \) will be called the pulse times. Furthermore, under the narrowband approximation [13] the time scale factor \( \alpha \) has a negligible effect on the pulse so that \( p(\alpha t) \approx p(t) \) so the model becomes

\[
r(t) = e^{j\tilde{\omega}t} e^{j\tilde{\phi}} \sum_{k=0}^{K-1} p(t - \tilde{T}_k).
\]  

(4)

The received signal is sampled at the Nyquist interval \( \Delta \) and the samples are assumed to be perturbed by complex zero-mean white Gaussian noise, \( \{w[n]\} \), with variance \( \sigma^2 \), resulting in

\[
x[n] = r[n] + w[n]
\]

\[
= e^{j\tilde{\omega}n} e^{j\tilde{\phi}} \sum_{k=0}^{K-1} p(\Delta n - \tilde{T}_k) + w[n].
\]  

(5)

From this received \( x[n] \) we wish to estimate the parameter vector

\[
\theta = \begin{bmatrix} \tilde{\omega} & \tilde{\phi} & \tilde{T}_0 & \tilde{T}_1 & \cdots & \tilde{T}_{K-1} \end{bmatrix}.
\]  

(6)

The key parameter to be estimated is the Doppler frequency \( \tilde{\omega} \), and the remaining unknowns are nuisance parameters that must be factored into the analysis (at least until shown irrelevant). It is recognized that the pulse, \( p(t) \), is generally not known, either; however, the CRLB will depend on the form of \( p(t) \) so numerical results for the CRLB will depend on the specific pulse shape assumed to have been intercepted.

**B. Derivation of the General Case FIM and CRLB for Doppler Frequency**

In preparation for computation of the CRLB [12], the derivatives of the signal with respect to each unknown are found to be

\[
\frac{\partial}{\partial \tilde{\omega}} r[n] = j\Delta nr[n]
\]  

(7)

\[
\frac{\partial}{\partial \tilde{\phi}} r[n] = jr[n]
\]  

(8)
\[
\frac{\partial}{\partial T_i} r[n] = -e^{j\Delta n} e^{j\phi} \sum_{k=0}^{K-1} \delta[k-l] |p'(\Delta n - T_k)| \\
= -e^{j\Delta n} e^{j\phi} |p'(\Delta n - T_i)|,
\]

where

\[
p'(\Delta n - T_i) = \frac{dp(t)}{dt} |_{t=\Delta n - T_i}.
\]

In order to simplify the expressions in the following derivation, define:

\[
S_0 = \frac{1}{N} \sum_{n=0}^{N-1} |p(n\Delta)|^2 \\
S_1 = \frac{1}{N} \sum_{n=0}^{N-1} n |p(n\Delta)|^2 \\
S_2 = \frac{1}{N} \sum_{n=0}^{N-1} n^2 |p(n\Delta)|^2
\]

\[
C_0 = \text{Im} \left\{ \frac{1}{N} \sum_{n=0}^{N-1} \Delta n p'(\Delta n) p''(\Delta n) \right\} \\
C_1 = \text{Im} \left\{ \frac{1}{N} \sum_{n=0}^{N-1} n p'(\Delta n) p''(\Delta n) \right\}
\]

\[
B = \frac{1}{S_0 N} \sum_{n=0}^{N-1} |p'(n\Delta)|^2
\]

\[
R_1 = \frac{1}{K} \sum_{k=0}^{K-1} \tilde{T_k} \\
R_2 = \frac{1}{K} \sum_{k=0}^{K-1} (\tilde{T_k})^2,
\]

where \(S_0\) is a measure of the pulse power, \(S_1\) is a measure of the pulse temporal centroid, \(S_2\) is a measure of the pulse temporal spread, \(C_0\) and \(C_1\) are time-frequency cross-coupling (or skew) measures, \(B\) is a measure of the pulse bandwidth, \(R_1\) is the average of the pulse times, and \(R_2\) is a measure of the temporal spread of the pulse arrival times. We also define \(I_k\) to be the index set where the \(k\)th received pulse samples are non-zero.

The elements of the Fisher Information Matrix (FIM), \(\mathbf{J}\), under the signal plus complex WGN assumption are given by
\[
J_{\phi\phi} = \frac{2}{\sigma^2} \text{Re} \left\{ \sum_n \left( \frac{\partial}{\partial \phi} r[n] \right) \left( \frac{\partial}{\partial \phi} r[n] \right)^* \right\} \\
= \frac{2}{\sigma^2} \sum_{k=0}^{K-1} \sum_{n \in I_k} (\Delta n)^2 |p(\Delta n - \bar{T}_k)|^2 \\
= \frac{2}{\sigma^2} \sum_{k=0}^{K-1} \sum_{n=0}^{N-1} (\Delta n + \bar{T}_k)^2 |p(\Delta n)|^2 \\
= \frac{2N}{\sigma^2} \left[ \Delta^2 KS_2 + 2\Delta KR S_1 + KR S_0 \right],
\]

where we have assumed that each pulse has \(N\) non-zero samples. Continuing in the same fashion gives

\[
J_{\phi\phi} = \frac{2}{\sigma^2} \text{Re} \left\{ \sum_n \left( \frac{\partial}{\partial \phi} r[n] \right) \left( \frac{\partial}{\partial \phi} r[n] \right)^* \right\} \\
= \frac{2}{\sigma^2} \sum_{k=0}^{K-1} \sum_{n=0}^{N-1} |p(\Delta n)|^2 = \frac{2}{\sigma^2} KNS_\phi,
\]

and

\[
J_{\phi\bar{\phi}} = \frac{2}{\sigma^2} \text{Re} \left\{ \sum_n \left( \frac{\partial}{\partial \bar{\phi}} r[n] \right) \left( \frac{\partial}{\partial \bar{\phi}} r[n] \right)^* \right\} \\
= \frac{2}{\sigma^2} \sum_{k=0}^{K-1} \sum_{n \in I_k} \Delta n \left| p(\Delta n - \bar{T}_k) \right|^2 \\
= \frac{2}{\sigma^2} \left[ \Delta KNS_\phi + NK R S_1 \right],
\]

and

\[
J_{\phi\bar{f}_k} = \frac{2}{\sigma^2} \text{Re} \left\{ \sum_n \left( \frac{\partial}{\partial \bar{\phi}} r[n] \right) \left( \frac{\partial}{\partial \bar{T}_k} r[n] \right)^* \right\} \\
= \frac{2}{\sigma^2} \text{Re} \left\{ \sum_{n \in I_k} -j \Delta n \ p(\Delta n - \bar{T}_k) \ p^*(\Delta n - \bar{T}_k) \right\} \\
= \frac{2}{\sigma^2} \text{Im} \left\{ \sum_{n=0}^{N-1} (\Delta n + \bar{T}_k) \ p(\Delta n) \ p^*(\Delta n) \right\} \\
= \frac{2N}{\sigma^2} \left[ \Delta C_1 + \bar{T}_k C_0 \right],
\]

and
\[ J_{\phi_{f_k}} = \frac{2}{\sigma^2} \text{Re} \left\{ \sum_n \left( \frac{\partial}{\partial \phi} r[n] \right) \left( \frac{\partial}{\partial T_k} r[n] \right)^* \right\} \]
\[ = \frac{2}{\sigma^2} \text{Re} \left\{ \sum_{n \neq f_k} -jp(\Delta n - \tilde{T}_k) \rho''(\Delta n - \tilde{T}_k) \right\} \]
\[ = \frac{2N}{\sigma^2} C_0, \]

and

\[ J_{t_{f_k}} = \frac{2}{\sigma^2} \text{Re} \left\{ \sum_n \left( \frac{\partial}{\partial T_k} r[n] \right) \left( \frac{\partial}{\partial T_k} r[n] \right)^* \right\} \]
\[ = \frac{2}{\sigma^2} NS_0 B, \]

and

\[ J_{\tilde{T}_{k+1}}|_{k \neq \ell} = \frac{2}{\sigma^2} \text{Re} \left\{ \sum_{n=-\infty}^{\infty} \left( \frac{\partial}{\partial T_k} r[n] \right) \left( \frac{\partial}{\partial T_{\ell}} r[n] \right)^* \right\} \]
\[ = 0, \]

where the last result is zero because the pulses are non-overlapping. These matrix elements form the FIM given by

\[
\begin{bmatrix}
\Delta^2 KS_2 + 2\Delta KR_i S_1 + KR_i S_0 & \Delta KS_i + KR_i S_0 & \Delta C_1 + \tilde{T}_0 C_0 & \Delta C_1 + \tilde{T}_1 C_0 & \cdots & \Delta C_1 + \tilde{T}_{K-1} C_0 \\
\Delta KS_i + KR_i S_0 & KS_0 & C_0 & C_0 & \cdots & C_0 \\
\Delta C_1 + \tilde{T}_0 C_0 & C_0 & S_0 B & 0 & \cdots & 0 \\
\Delta C_1 + \tilde{T}_1 C_0 & C_0 & 0 & S_0 B & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & 0 \\
\Delta C_1 + \tilde{T}_{K-1} C_0 & C_0 & 0 & \cdots & 0 & S_0 B
\end{bmatrix}
\]

This structure shows that in general there is correlation between the estimates of most of the parameters, except between the pulse times. The algorithm proposed in [6] relies on cross-correlation to estimate the pulse times;
thus, it is clear from the above FIM that the accuracy of those pulse time estimates will impact the overall accuracy of the Doppler estimate.

To proceed further, defining the partitions shown in (22) as

$$
J = \frac{2N}{\sigma^2} \begin{bmatrix} A & C \\ C^T & B \end{bmatrix},
$$

(23)

allows use of standard results for inverting symmetric partitioned matrices to yield the 2×2 CRLB matrix for Doppler frequency and phase:

$$
CRLB_{\omega, \phi} = \frac{\sigma^2}{2N} \left( A - CB^{-1}C^T \right)^{-1}
$$

$$
= \frac{\sigma^2}{2N} \left( A - \frac{CC^T}{S_0 B} \right)^{-1},
$$

(24)

where the fact that matrix B is a scalar ($S_0 B$) times the identity has been used. Evaluating the form for $CC^T$ and substituting into (24) gives

$$
CRLB_{\omega, \phi} = \frac{\sigma^2}{2N} \begin{bmatrix} \Delta^2 K\tilde{S}_2 + 2\Delta KR_1 \tilde{S}_1 + KR_2 \tilde{S}_0 & \Delta K\tilde{S}_1 + KR_1 \tilde{S}_0 \\ \Delta K\tilde{S}_1 + KR_1 \tilde{S}_0 & K\tilde{S}_0 \end{bmatrix} \begin{bmatrix} \Delta R_1 \tilde{S}_0 + 2\Delta R_1 \tilde{S}_0 + R_2 \tilde{S}_0 \\ \Delta R_1 \tilde{S}_0 + 2\Delta R_1 \tilde{S}_0 + R_2 \tilde{S}_0 \end{bmatrix}^{-1},
$$

(25)

with the following definitions

$$
\tilde{S}_0 \triangleq S_0 - (C_0^2 / S_0 B) \quad \tilde{S}_1 \triangleq S_1 - (C_0 C_1 / S_0 B) \quad \tilde{S}_2 \triangleq S_2 - (C_1^2 / S_0 B).
$$

(26)

Finding the 1,1 element of (25) gives the CRLB for the Doppler frequency estimate. Using Cramer’s rule gives

$$
CRLB_{\omega} = \frac{\sigma^2}{2NK} \frac{\tilde{S}_0}{(\Delta^2 \tilde{S}_2 + 2\Delta R_1 \tilde{S}_1 + R_2 \tilde{S}_0) - (\Delta \tilde{S}_1 + R_0 \tilde{S}_0)^2}
$$

$$
= \frac{\sigma^2}{2NK S_0} \frac{1}{\Delta^2 [(\tilde{S}_2 / \tilde{S}_0) - (\tilde{S}_1^2 / \tilde{S}_0^2)] + R_2 - R_1^2}
$$

$$
= \frac{1}{2NK \text{SNR}} \frac{1}{\text{SNR} \left[1 - \frac{C_0^2}{S_0^2} \right] (\Delta^2 \tilde{D} + R_2)},
$$

(27)

where $\text{SNR} \triangleq S_0 / \sigma^2$ and
\[ \tilde{D} \equiv \left( \frac{\hat{S}_2}{\hat{S}_0} \right) - \left( \frac{\hat{S}_1}{\hat{S}_0} \right)^2 \quad \tilde{R}_z \equiv R_z - R_i^2 = \frac{1}{K} \sum_{k=0}^{K-1} (\hat{T}_k - R_i)^2. \]  

(28)

Note that \( \tilde{D} \) is a “mean-normalized” measure of the pulse duration and \( \tilde{R}_z \) is a “mean-normalized” version of the time spread of the pulse times. These mean-normalized measures ensure that the choice of time origin has no effect on the CRLB for the Doppler estimate.

C. Special Cases for the FIM and CRLB

To allow some further insight into this result some special cases are now considered. Note that the values in the upper-right and lower-left corners of the FIM in (22) depend on the quantities \( C_0 \) and \( C_1 \), which are shown in (12) to be the imaginary part of an expression. Thus, if the pulse \( p(t) \) is such that the quantity inside the Im{} in (12) is purely real then many terms in (22) become zero and yields a block diagonal form given by

\[
\begin{bmatrix}
\Delta^2 K S_2 + 2 \Delta K R_1 S_1 + K R_2 S_0 & \Delta K S_1 + K R_1 S_0 & 0 & 0 & \cdots & 0 \\
\Delta K S_1 + K R_1 S_0 & K S_0 & 0 & 0 & \cdots & 0 \\
0 & 0 & S_0 B & 0 & \cdots & 0 \\
0 & 0 & 0 & S_0 B & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \cdots & 0 & S_0 B
\end{bmatrix}.
\]

(29)

This block diagonal structure gives a simple form and shows that for this special case the estimation of the frequency and phase is independent of the estimation of the pulse times. A sufficient (but not necessary) condition to get the FIM as in (29) is that \( p(t) \) be purely real valued, which will make both \( C_0 \) and \( C_1 \) in (12) equal to zero. This holds for a radar pulse train with no phase/frequency modulation on the pulses. Because of the block diagonal structure in (29), the CRLB for Doppler frequency can be found by inverting the 2×2 matrix in the upper-left corner of (29) and finding the 1,1 element of the result. Using Cramer’s rule gives
\[ CRLB_{\omega} = \frac{\sigma^2}{2NK} \frac{S_0}{S_0 \left( \Delta^2 S_2 + 2\Delta R S_1 + R_2 S_0 \right) - (\Delta S_1 + R_1 S_0)^2} \]

where \( D \equiv \left( \frac{S_2}{S_0} \right) - \left( \frac{S_1}{S_0} \right)^2 \) is a measure of pulse duration.

As mentioned above when we defined the mean-normalized measures \( \tilde{D} \) and \( \tilde{R}_2 \), the estimation of frequency is expected to be time-shift invariant. That is, the frequency estimation accuracy is not dependent on the time reference on which \( T_1, \ldots, T_K \) and the pulse’s starting point are based. Therefore, without loss of generality, we can assume that \( R_1 \) (the mean of the pulse times) is zero, and that \( S_1 \) (the pulse centroid) is also zero. It should be noted that to get \( S_1 = 0 \) we are now assuming that the pulse \( p(t) \) has support over an interval that extends over negative times as well as positive times; thus, the sums defining the \( S_0, S_1, \) and \( S_2 \) in (11) must now start at some negative integer. Note that this simplification does affect the estimate of \( \phi \), the phase offset, but it is of little consequence since it is a nuisance parameter in the first place. These simplifications, together with the earlier simplification that \( C_0 = C_1 = 0 \), lead to

\[ J = \frac{2N}{\sigma^2} \text{diag} \left\{ \Delta^2 K S_2, K R_2 S_0, K S_0, S_0 B, \ldots, S_0 B \right\}, \] (31)

which decouples the phase and frequency estimation. The CRLB matrix is now easily found to be

\[ CRLB = \frac{\sigma^2}{2NS_0} \text{diag} \left\{ \frac{1}{\Delta^2 K \left( \frac{S_2}{S_0} \right) + KR_2 \frac{1}{K} B}, \frac{1}{1 B}, \ldots, \frac{1}{1 B} \right\}, \] (32)

where, as previously defined, \( K \) is the number of pulses, \( N \) is the number of samples per pulse, \( B \) is a measure of pulse bandwidth defined in (13), \( D \) is a measure of pulse duration defined after (30), \( R_2 \) is a measure of spread of the pulse arrival times defined in (14), and \( \Delta \) is the sampling interval in seconds. Because of the diagonal nature of this matrix, for this special case the lack of knowledge on the nuisance parameters does not impact the frequency estimate. The CRLB for frequency estimation for this special case is
\[ CRLB_\delta = \frac{1}{2NK \text{SNR}\left(\Delta^2 S_2 / S_0 + R_2\right)}. \] (33)

There are four key parts contributing to the denominator: the SNR, a measure of pulse duration \((D)\), a measure of pulse spread \((R_2)\), and the total number of samples \((NK)\).

If there is only one pulse – again assuming that \(p(t)\) is such that \(C_0 = C_1 = 0\) and \(S_1 = 0\) – then the single pulse time is \(\bar{T}_0 = 0\) and (33) becomes

\[ CRLB_\delta = \frac{1}{2N \text{SNR}\Delta^2 (S_2 / S_0)}. \] (34)

Similarly, for the case of a single rectangular pulse of amplitude \(A\) with no phase/frequency modulation that starts at \(t = 0\) with known \(\bar{T}_0 = 0\); then \(R_1 = R_2 = 0\) and the FIM in (22) becomes

\[ J = \frac{2N}{\sigma^2} \begin{bmatrix} \Delta^2 S_2 & \Delta S_1 \\ \Delta S_1 & S_0 \end{bmatrix} = \frac{2A^2}{\sigma^2} \begin{bmatrix} \Delta^2 \sum_{n=0}^{N-1} n^2 & \Delta \sum_{n=0}^{N-1} n \\ \Delta \sum_{n=0}^{N-1} n & N \end{bmatrix}, \] (35)

and the Cramer-Rao bound in (30) becomes

\[ CRLB_\delta = \frac{6}{\Delta^2 \text{SNR} N(N^2 - 1)}, \] (36)

where \(\text{SNR} \triangleq A^2 / \sigma^2\); these are exactly the well-known results for the FIM and CRLB for the estimation of frequency and phase for a complex sinusoid in white Gaussian noise [14]. Thus, the results given here for the pulsed case are consistent with known results for estimating the frequency of a single complex sinusoid.

### III. CRLB for Frequency Estimation of Non-Coherent Pulse Train

As a comparison we consider the case of a non-coherent pulse train from an emitter, consisting of \(K\) pulses where the phase for each pulse is arbitrary (i.e. no specific relationship),

\[ s(t) = e^{j\omega_0} \sum_{k=0}^{K-1} e^{j\phi_k} p(t - T_k). \] (37)
Following the same steps as before yields the received data samples

\[ x[n] = r[n] + w[n] \]

\[ = e^{j\omega n} \sum_{k=0}^{K-1} e^{j\phi_k} p(\Delta n - \bar{T}_i) + w[n]. \]  

(38)

The unknown phases give \( K \) additional parameters to be estimated; the parameter vector is now

\[ \theta = [\tilde{\omega}, \tilde{\phi}_0, \tilde{\phi}_1, \ldots, \tilde{\phi}_{K-1}, \bar{T}_0, \bar{T}_1, \ldots, \bar{T}_{K-1}]^T. \] 

(39)

This section explores this scenario only for the simplified case where the pulse \( p(t) \) is such that (i) the expressions in (12) evaluate to \( C_0 = C_1 = 0 \) and (ii) the expression for \( S_1 \) in (11) evaluates to \( S_1 = 0 \).

Results for the more general case could be derived following steps similar to those in Section II-B but since the non-coherent method is known to not provide sufficient accuracy for location, these results are presented only to allow understanding of the impact of coherency and consideration of the special case is sufficient to achieve this insight.

Following steps similar to those for the coherent pulse train case gives

\[ J_{\omega\omega} = \frac{2N}{\sigma^2} \left[ \Delta^2 K S_2 + K R S_0 \right], \]

\[ J_{\phi_k\phi_l} = \frac{2N}{\sigma^2} \bar{T} S_0, \]

\[ J_{\bar{T}\bar{T}} = \frac{2N}{\sigma^2} \bar{T} S_0, \]

\[ J_{\phi_k\bar{T}} = \frac{2N}{\sigma^2} \bar{T} S_0. \] 

(40)

\[ J_{\omega\phi_k} = 0 \]

\[ J_{\phi_k\phi_l} \bigg|_{k \neq l} = 0 \]

\[ J_{\phi_k\bar{T}} = 0 \]

\[ J_{\bar{T}\bar{T}} \bigg|_{k \neq l} = 0 \]

\[ J_{\bar{T}\bar{T}} \bigg|_{k = l} = 0. \] 

(41)

The resulting FIM is then
This is block diagonal, so inverting the \((K+1) \times (K+1)\) matrix in the upper-left corner of (42), which can be done using again the standard result for partitioned symmetric matrices gives

\[
\begin{bmatrix}
\Delta^2 KS_2 + KR_2 S_0 & S_0 \tilde{T}_0 & S_0 \tilde{T}_1 & \cdots & S_0 \tilde{T}_{K-1} \\
S_0 \tilde{T}_0 & S_0 & 0 & \cdots & 0 \\
S_0 \tilde{T}_1 & 0 & S_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
S_0 \tilde{T}_{K-1} & 0 & \cdots & 0 & S_0 \\
0 & 0 & 0 & \cdots & 0 & S_0 B \\
0 & 0 & 0 & \cdots & 0 & 0 & S_0 B \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & S_0 B
\end{bmatrix}.
\]

(42)

Comparing the non-coherent case result in (43) to the coherent case results in (33) shows that the non-coherent case result lacks the effect of the pulse spread term \(R_2\) that gets added to \(\Delta^2 D\) to decrease the CRLB for the coherent case. In fact, the \(K\)-pulse non-coherent result in (43) is simply \(1/K^2\) the bound for a single pulse; thus, the non-coherent case is equivalent to processing each pulse individually to estimate the frequency and then averaging the \(K\) resulting pulse-local estimates together. That is, for the non-coherent pulse train there is no ability to exploit any pulse-to-pulse structure – as should be expected. The gain in the coherent case comes from the fact that we can exploit pulse-to-pulse structure because of the fixed phase relationship between the pulses.
IV. Discussion

Note that in all the CRLB results given above the CRLB depended on $N\Delta^2$; to provide proper physical insight it is best that one of the $\Delta$s in the $\Delta^2$ term be grouped with the $N$ to form the pulse’s “total on-time” $N\Delta = T_{OR}$; note that for a fixed-length pulse the value of $N\Delta$ remains fixed regardless of the value of $\Delta$. In addition, the terms $S_2/S_0$, $D$, and $\hat{D}$ are in units of samples$^2$ and it makes sense to multiply those by $\Delta^2$ to convert into a quantity in units of seconds$^2$; this is done by multiplying by inserting additional $\Delta$ that is then neutralized by multiplying by the sampling rate $F_s$.

The results obtained above (modified as just mentioned) are summarized in Table 1. Notice also that all the CRLB results in Table 1 depend inversely on $2T_{OR} SNR F_s$. This dependence shows that doubling the pulse on-time $T_{OR}$ halves the CRLB. It may also appear that doubling the sampling rate $F_s$ will also halve the CRLB but this is likely not achievable as discussed in the following. One should take care to remember that increasing $F_s$ without increasing the front-end bandwidth will make the noise samples non-white, so to apply these white-noise results would require a commensurate increase in front-end bandwidth, which will increase the noise power but not increase the signal power. The resulting decrease in $SNR$ perfectly counters the increase in $F_s$ so that the product $SNR F_s$ remains the same. Changes in $F_s$ will cause changes in $\Delta^2 D$ (and other similar terms) but those changes will be small for all $F_s$ above the Nyquist rate. Thus, we see that there should be very little change as a function of sampling rate, which makes sense in terms of sampling theory that says that sampling faster than the Nyquist rate provides no additional information about the signal.

The last column in Table 1 lists definitions used in the results; some definitions used in a given row’s result may have been stated in a higher row. Also note that all sums over $n$ are shown without a specific range to allow choice of the starting time of pulse $p(t)$, which affects the value of $S_1$.

The first row considers the case of a single pulse with zero temporal centroid ($S_1 = 0$) where the result shows that the CRLB is inversely proportional to $S_2/S_0$, which is a classic measure [13] of a zero-centroid pulse’s temporal dispersion. For a zero-centroid pulse the quantity $\sqrt{S_2/S_0}$ is called the Gabor duration or
RMS duration [13] and here has units of samples; the quantity $\Delta \sqrt{S_i / S_0}$ has units of seconds. The division by $S_0$ normalizes out the effect of pulse power on this measure. Thus, we see that the CRLB for the single pulse case depends on two separate measures of pulse duration: pulse on-time $T_{OT}$ and Gabor duration $\sqrt{S_i / S_0}$. The second row also considers the single pulse case but now allows that $S_i \neq 0$. The only difference here is that $S_i / S_0$ is replaced by $D$ as defined in the second row’s Definitions column; note that the second line in the definition for $D$ shows that it subtracts $S_i / S_0$ to essentially center time axis to yield a zero-centroid condition. Thus, this result is an obvious and consistent extension of the first row.

The third row gives the results for a non-coherent pulse train of $K$ pulses under the simplifying conditions that $C_0 = C_1 = S_1 = 0$. This result was derived for the case of $R_1 = 0$ but the following discussion shows that it holds also for $R_1 \neq 0$. Note that the definition of $R_1$ shows that it is the average pulse time, so $R_1 = 0$ means that the entire pulse train is centered in some sense at the time origin. Again we see the same dependence on $S_i / S_0$ as in the first row; note that if $S_i \neq 0$ the quantity $S_i / S_0$ can be replaced by $D$. Thus, the only difference between a single pulse and a $K$-pulse non-coherent pulse train is that the CRLB is reduced by the division by $K$; that is, an estimate is obtained from each pulse and then the $K$ estimates are averaged. It is clear then that this result holds even if $R_1 \neq 0$.

The fourth row gives a simple coherent $K$-pulse case to compare with the simple non-coherent case in the third row. Here the CRLB depends on a new factor, $R_2$, that is a measure of the temporal spread of the pulse train in units of seconds$^2$. Strictly, $R_2$ is a mean-square value of the pulse times, but since here the average pulse time is $R_1 = 0$ the value of $R_2$ is really a variance of the pulse times. Comparing the simplified-case results for non-coherent and coherent pulse trains (rows 3 and 4, respectively) shows that the gain due to coherency comes from the ability to exploit the spread between pulses that is captured by $R_2$: more widely spaced pulses increases $R_2$ which adds to $S_i / S_0$ to decrease the CRLB. Because $\Delta^2 S_i / S_0$ is a measure of pulse duration and $R_2$ is a measure of pulse train duration, the value of $R_2$ will be much larger than $\Delta^2 S_i / S_0$ and that leads to a significant decrease in the CRLB. The approximation shown relies on this dominance of $R_2$. 
The fifth row shows the case when $S_1 \neq 0$ and $R_1 \neq 0$, where comparing to the fourth row shows that this is handled by replacing $S_2/S_0$ by $D$ and $R_2$ by $\tilde{R}_2$. Just as $D$ was discussed as a centroid-normalized version of $S_2$, $\tilde{R}_2$ is a centroid-normalized version of $R_2$; that is, $\tilde{R}_2$ is the variance of the pulse times about the mean pulse time.

Finally, the sixth row shows the most general result for the coherent pulse train case, where some of the previously defined quantities are adjusted by the effect of so-called skew measures [13] given by $C_0$ and $C_1$. There are two main effects: (i) the duration as measured by $\tilde{D}$ is reduced from $D$ through the subtraction of skew term in the definition of $\tilde{S}_2 = S_2 - (C_1^2 / S_0 B)$, and (ii) the effective $SNR$ is reduced by multiplication of the term $\left[1 - (C_0^2 / \tilde{S}_0^2 B)\right]$. As discussed above, though, the impact of $\tilde{D}$ is negligible compared to $\tilde{R}_2$ so the effect of the skew parameters via $\tilde{D}$ can be ignored, as shown in the first approximation in the sixth row. Thus, $C_1$ has negligible impact and the only potentially significant impact of the skew is via $\left[1 - (C_0^2 / \tilde{S}_0^2 B)\right]$. However, note that the impact here of $C_0$ is reduced when the value of $B$ is large; note that the derivatives in the definition of $B$ can be converted into the frequency domain to give a measure of bandwidth [12]. Thus, phase/frequency modulation will tend to introduce the impact of the skew parameters but the increased bandwidth due to the modulation will temper its impact. In fact, it is possible to show (see the Appendix) that the factor $C_0$ will be zero if the instantaneous frequency of the pulse $p(t)$ is symmetric (as for a linear FM pulse) and in other cases is expected to be small enough to be negligible; thus, the second approximation in the sixth row is expected to hold for most if not all pulse trains.

Comparing all the final results (i.e., after approximations) for the coherent case in Table 1 shows that as a general rule of thumb the CRLB depends inversely on: (i) the pulse on-time $T_{OT}$, (ii) the number of pulses $K$, (iii) the variance of pulse times $\tilde{R}_2$, and (iv) the product $SNR \times F_s$; the pulse shape and pulse modulation seem to have little or no impact on the frequency accuracy. As a special case consider the scenario where the intercepted pulses are equally spaced by $T_{PRI}$, the pulse repetition interval (PRI), then it is easily shown that $\tilde{R}_2 = T_{PRI}^2 (K^2 - 1) / 12$ and the last approximation in the sixth row of Table 1 can be re-written as
\[ CRLB_{\text{Coh}} \approx \frac{6}{T_{\text{OT}} K (K^2 - 1) \text{SNR} F_s T_{\text{PRI}}^2}, \]  
\hspace{1cm} (44) 

from which we see that the frequency CRLB varies as \( O(1 / T_{\text{PRI}}^2) \) and as \( O(1 / K^3) \). Thus, for all other parameters fixed, doubling the number of equally spaced pulses decreases the CRLB by \( 1/8 \).

It is interesting to make a comparison between the coherent and non-coherent cases, which will be done here for the case of rectangular pulses spaced by the PRI. Expressing the non-coherent case CRLB similarly gives, using \( 1/K^d \) the result in (36), gives

\[ CRLB_{\text{Non-Coh}} = \frac{6}{T_{\text{OT}} K \text{SNR} F_s \Delta^2 \left( N^2 - 1 \right)} \]  
\hspace{1cm} (45) 

together with (44) and taking that coherent gain in accuracy as the ratio

\[ \text{Gain} = \frac{\sqrt{CRLB_{\text{Non-Coh}}}}{\sqrt{CRLB_{\text{Coh}}}} \approx \frac{K T_{\text{PRI}}}{T_{\text{OT}}}, \]  
\hspace{1cm} (46) 

where the following approximations have been used: \( T_{\text{OT}}^2 \approx \Delta^2 \left( N^2 - 1 \right) \) and \( K^2 - 1 \approx K^2 \). This gives a rule of that that the gain in coherent accuracy is \( K \) times the ratio of pulse on-time to PRI; since the PRI is typically much larger than the pulse on-time there is great improvement available when one can exploit the coherency of the pulse train.

Numerically computing the CRLB for typical values of pulse train parameters shows that it is possible to achieve frequency accuracy much lower than 1 Hz (1 Hz is the accuracy that is claimed experimentally in [4] and [5]; the accuracy for the simulation results in [11] are on the order of a few Hz). Thus, it is clear that the existing algorithms, while achieving quite good accuracies, are still far from achieving the CRLB. Numerical results are given in Table 2 for several typical cases of pulse on-time, pulse shape, PRI, and number of pulses. In all cases the pulse \( p(t) \) was real-valued and had a shape given by the Tukey window [15]; the column in Table 2 labeled \( \alpha \) is the taper parameter [15] that is used here to control the shape of the pulse and hence the value of \( \Delta \sqrt{D} \) (note that \( \alpha = 0 \) corresponds to zero taper and so is a rectangular pulse; \( \alpha = 1 \) corresponds to full taper and has no flat region in the center). In Table 2 the CRLB for the coherent case is computed using
\[ CRLB_{\omega} = \frac{1}{2T_{ot} K \text{SNR} F} \left[ \Delta^2 D + \hat{R}_2 \right], \quad (47) \]

while for the non-coherent case it is computed using

\[ CRLB_{\omega} = \frac{1}{2T_{ot} K \text{SNR} F} \Delta^2 D; \quad (48) \]

the values for the Cramer-Rao bounds are tabulated in square-root form in units of Hz to represent a bound on frequency estimation standard deviation in Hz. In addition, Table 2 lists the computed values for \( \Delta \sqrt{D} \) and \( \sqrt{\hat{R}_2} \) to illustrate the relationship between them; note that \( \sqrt{\hat{R}_2} \) always dominates and therefore the coherent CRLB does not depend on \( \Delta \sqrt{D} \).

For comparison, Table 3 presents some simulation accuracy results published in [11] side-by-side with the corresponding computed coherent CRLB results. It should be noted that the results in [11] were given for the case of staggered PRI values within each pulse train, although the computed CRLB results given in Table 3 were computed without the stagger for convenience; the stagger was slight and would not significantly change the computed CRLB values. Note that the CRLB values are much smaller than the reported simulation accuracies; also note that the discrepancy between simulation results and CRLB are worse at higher SNR. In fact, in light of the dependence on SNR shown in (47) we would expect the ratio of \( \sqrt{CRLB} \) at 37 dB relative to at 23 dB to be 5; however, as seen in Table 3 the simulation results exhibit ratios of only 1.4 to 2.8. This form of dependence on SNR is characteristic of both coherent and non-coherent frequency estimation methods as well as the classic frequency estimation methods. This indicates that although the algorithm in [11] produces very accurate methods that its SNR-behavior does not follow that of the CRLB. Looking at the ratio of the \( \sqrt{CRLB} \) at 4 pulses relative to at 6 pulses we expect the ratio to be 1.87 and the simulation results exhibit ratios from 1.4 to 2.1; indicating that the algorithm’s behavior relative to pulse number follows the CRLB behavior more closely than for the SNR-behavior.
### Table 1: Summary of Results for CRLB on Frequency of Pulse Train

<table>
<thead>
<tr>
<th>Case</th>
<th>CRLB Result ((rad/sec)²)</th>
<th>Definitions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>#1: Single Pulse Case with $S_1 = 0$</strong></td>
<td>$\text{CRLB}<em>0 = \frac{1}{2T</em>{\text{OT}} \text{ SNR} F_s \Delta^2 (S_2 / S_0)}$</td>
<td>$S_0 = \frac{1}{N} \sum_n</td>
</tr>
<tr>
<td><strong>#2: Single Pulse Case</strong></td>
<td>$\text{CRLB}<em>0 = \frac{1}{2T</em>{\text{OT}} \text{ SNR} F_s \Delta^2 D}$</td>
<td>$D = (S_2 / S_0) - (S_i / S_0)^2$ \quad $\frac{1}{N} \sum_n (n - (S_i / S_0))^2</td>
</tr>
<tr>
<td><strong>#3: Non-Coherent Case with $C_0 = C_1 = 0$</strong></td>
<td>$\text{CRLB}<em>0 = \frac{1}{2T</em>{\text{OT}} K \text{ SNR} \Delta^2 (S_2 / S_0)} + R_2$</td>
<td>$K$ = # of pulses \quad $C_0 = \text{Im} \left{ \frac{1}{N} \sum_n p(\Delta n) p^<em>(\Delta n) \right}$ \quad $C_1 = \text{Im} \left{ \frac{1}{N} \sum_n n p(\Delta n) p^</em>(\Delta n) \right}$ \quad $R_1 = 1 / K \sum K_{K=0}^{K-1} \bar{T}_k$ \quad $\bar{T}_k = \text{pulse times}$</td>
</tr>
<tr>
<td><strong>#4: Coherent Case with $C_0 = C_1 = 0$</strong></td>
<td>$\text{CRLB}<em>0 = \frac{1}{2T</em>{\text{OT}} K \text{ SNR} F_s \left[ \Delta^2 (S_2 / S_0) + R_2 \right]} + R_2$</td>
<td>$R_2 = 1 / K \sum K_{K=0}^{K-1} \bar{T}_k$ \quad $\bar{T}_k = \text{pulse times}$</td>
</tr>
<tr>
<td><strong>#5: Coherent Case with $C_0 = C_1 = 0$</strong></td>
<td>$\text{CRLB}<em>0 = \frac{1}{2T</em>{\text{OT}} K \text{ SNR} F_s \left[ \Delta^2 D + \bar{R}_2 \right]}$</td>
<td>$R_1 = 1 / K \sum K_{K=0}^{K-1} \bar{T}_k$ \quad $\bar{R}<em>2 = R_2 - R_2^2 = 1 / K \sum K</em>{K=0}^{K-1} \left( \bar{T}_k - R_1 \right)^2$</td>
</tr>
<tr>
<td><strong>#6: General Coherent Case</strong></td>
<td>$\text{CRLB}<em>0 = \frac{1}{2T</em>{\text{OT}} K \text{ SNR} F_s \left[ \Delta^2 D + \bar{R}_2 \right]}$ \quad \quad $B = 1 / S_0 \sum N_n</td>
<td>p'(n\Delta)</td>
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### Table 2: Numerical Results for $F_s = 100$ MHz and SNR = 10 dB

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<th>$T_{PRI}$ (ms)</th>
<th>$K$</th>
<th>$T_{OT}$ (µs)</th>
<th>$\alpha$</th>
<th>$\Delta \sqrt{D}$ (µs)</th>
<th>$\sqrt{R_2}$ (ms)</th>
<th>Non-Coherent $\sqrt{\text{CRLB}_{\omega}} / 2\pi$ (Hz)</th>
<th>Coherent $\sqrt{\text{CRLB}_{\omega}} / 2\pi$ (Hz)</th>
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<td>$\sigma_j$ in [11] (Hz)</td>
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<td>37</td>
<td>0.08</td>
<td>0.9</td>
<td>11.8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Appendix: Evaluation of $C_0$

The CRLB for the most general coherent pulse train case was shown to depend on the value of $C_0$ given in (12), which is evaluated here using an integral approximation of the summation given by

$$C_0 = \frac{1}{N} \text{Im} \left\{ \sum_n p(\Delta n) p^* (\Delta n) \right\} \approx \frac{1}{N} \text{Im} \left\{ \frac{T_{\text{GR}}}{\Delta} \int_0^{T_{\text{GR}}} p(t)p^*(t) dt \right\}. \quad (49)$$

As discussed in the main body of the paper, $C_0$ is non-zero only when $p(t)$ is complex, where it can be expressed in polar form or rectangular form, both of which are shown here as

$$p(t) = A(t)e^{j\phi(t)} = A(t)\cos[\phi(t)] + jA(t)\sin[\phi(t)], \quad (50)$$

where $A(t)$ is the real-valued envelope function and $\phi(t)$ is the real-valued phase function. Then

$$C_0 \approx \frac{1}{N} \text{Im} \left\{ \frac{T_{\text{GR}}}{\Delta} \int_0^{T_{\text{GR}}} [a(t) + jb(t)][a'(t) - j b'(t)] dt \right\}, \quad (51)$$

$$= \frac{1}{N} \frac{T_{\text{GR}}}{\Delta} \int_0^{T_{\text{GR}}} [a'(t)b(t) - a(t)b'(t)] dt.$$ 

But, using (50) gives

$$a'(t) = A'(t)\cos[\phi(t)] - A(t)\phi'(t)\sin[\phi(t)]$$

$$b'(t) = A'(t)\sin[\phi(t)] + A(t)\phi'(t)\cos[\phi(t)] \quad (52)$$

and

$$a'(t)b(t) = A(t)\sin[\phi(t)][A'(t)\cos[\phi(t)] - A(t)\phi'(t)\sin[\phi(t)]]$$

$$= A(t)A'(t)\sin[\phi(t)]\cos[\phi(t)] - A(t)\phi'(t)\sin^2[\phi(t)]$$

$$a(t)b'(t) = A(t)\cos[\phi(t)][A'(t)\sin[\phi(t)] + A(t)\phi'(t)\cos[\phi(t)]]$$

$$= A(t)A'(t)\sin[\phi(t)]\cos[\phi(t)] + A(t)\phi'(t)\cos^2[\phi(t)] \quad (53)$$

which when substituted into (51) gives
\[
C_0 \approx -\frac{1}{N} \frac{1}{\Delta} \int_0^T A^2(t)\phi'(t)dt.
\] (54)

Recognizing that \(\phi'(t) = \omega(t)\), the instantaneous frequency, we can write

\[
C_0 \approx -\frac{1}{N} \frac{1}{\Delta} \int_0^T A^2(t)\omega(t)dt.
\] (55)

For any signal for whom the instantaneous frequency varies symmetrically around the carrier frequency (i.e., the frequency to be estimated) then (55) implies that \(C_0\) will be zero; this includes linear FM pulses. It is also approximately true for pulses that are subjected to BPSK with a large enough number of phase transitions within each pulse. Although it is difficult to make a wide-sweeping general conclusion here, it is also likely that most if not all typical phase modulations will give a small value of \(C_0\) because the instantaneous frequency typically varies uniformly (at least approximately) above and below zero. Also, it should be noted that the effect of any non-zero value of \(C_0\) will be deemphasized through the division in \([1 - (C_0^2 / S_0^2 B)]\). Thus, we conjecture that the parameter \(C_0\) has little effect.

References


