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EECE 301
Signals & Systems
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Discussion #3a

- Review of Differential Equations

Differential Equations Review

Differential Equations like this are Linear and Time Invariant:

$$\boxed{a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_0 y(t) = b_m \frac{d^m f(t)}{dt^m} + \ldots + b_1 \frac{df(t)}{dt} + b_0 f(t)}$$

-coefficients are constants $\Rightarrow$ **TI**

-No nonlinear terms $\Rightarrow$ **Linear**

Examples of Nonlinear Terms:

$$f^n(t), \quad \left[\frac{d^k y(t)}{dt^k}\right]\left[\frac{d^p y(t)}{dt^p}\right], \quad y^n(t), \quad \left[\frac{d^k y(t)}{dt^k}\right]\left[\frac{d^p y(t)}{dt^p}\right], \quad etc.$$

In the following we will BRIEFLY review the basics of solving Linear, Constant Coefficient Differential Equations under the Homogeneous Condition

“Homogeneous” means the “forcing function” is zero

That means we are finding the “zero-input response” that occurs due to the effect of the initial coniditions.

We will assume: $m \leq n$

m is the highest-order derivative on the “input” side

n is the highest-order derivative on the “output” side

Use “operational notation”: $\boxed{\dfrac{d^k y(t)}{dt^k} \equiv D^k y(t)}$

$\Rightarrow$ Write D.E. like this:

$$\underbrace{\left(D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0\right)}_{\triangleq Q(D)} y(t) = \underbrace{\left(b_m D^m + \ldots + b_1 D + b_0\right)}_{\triangleq P(D)} f(t)$$

$$Diff.\ Eq. \Rightarrow \quad \boxed{Q(D)y(t) = P(D)f(t)}$$

<u>Due to linearity:</u> Total Response = Zero-Input Response + Zero-State Response

<u>Z-I Response</u>: found assuming the input $f(t) = 0$ but with given IC's

<u>Z-S Response</u>: found assuming IC's = 0 but with given $f(t)$ applied

<u>Finding the Zero-Input Response</u> (Homogeneous Solution)

Assume $f(t) = 0$

(▲)

$$\Rightarrow D.E.: Q(D)y_{zi}(t) = 0$$

$$\Rightarrow \underbrace{\left(D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0\right)y_{zi}(t)} = 0 \qquad \forall t > 0$$

"linear combination" of $y_{zi}(t)$ & its derivatives must be = 0

Consider $y_0(t) = ce^{\lambda t}$

c and λ are possibly complex numbers

Can we find c and λ such that $y_0(t)$ qualifies as a homogeneous solution?

Put $y_0(t)$ into (▲) and use result for the derivative of an exponential:

$$\boxed{\frac{d^n e^{\lambda t}}{dt^n} = \lambda^n e^{\lambda t}}$$

$$c\underbrace{(\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0)}_{\text{must} = 0}e^{\lambda t} = 0$$

Characteristic polynomial

$Q(\lambda)$ has at most n unique roots

(can be complex)

$$\Rightarrow Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$$

$c_1 e^{\lambda_1 t}$ is a solution

$c_2 e^{\lambda_2 t}$ is a solution

$\vdots$

$c_n e^{\lambda_n t}$ is a solution

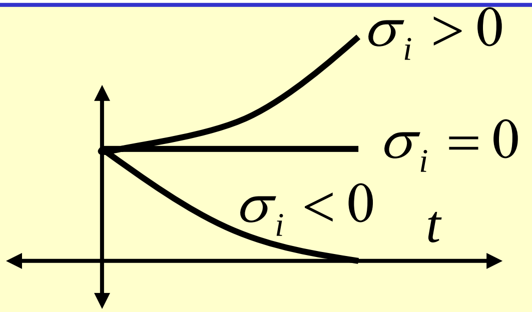
So…any linear combination
is also a solution to (▲)

$$\boxed{\text{Z-I Solution}: y_{zi}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \ldots + c_n e^{\lambda_n t}}$$

Then, choose $c_1, c_2, \ldots, c_n$ to satisfy the given IC's

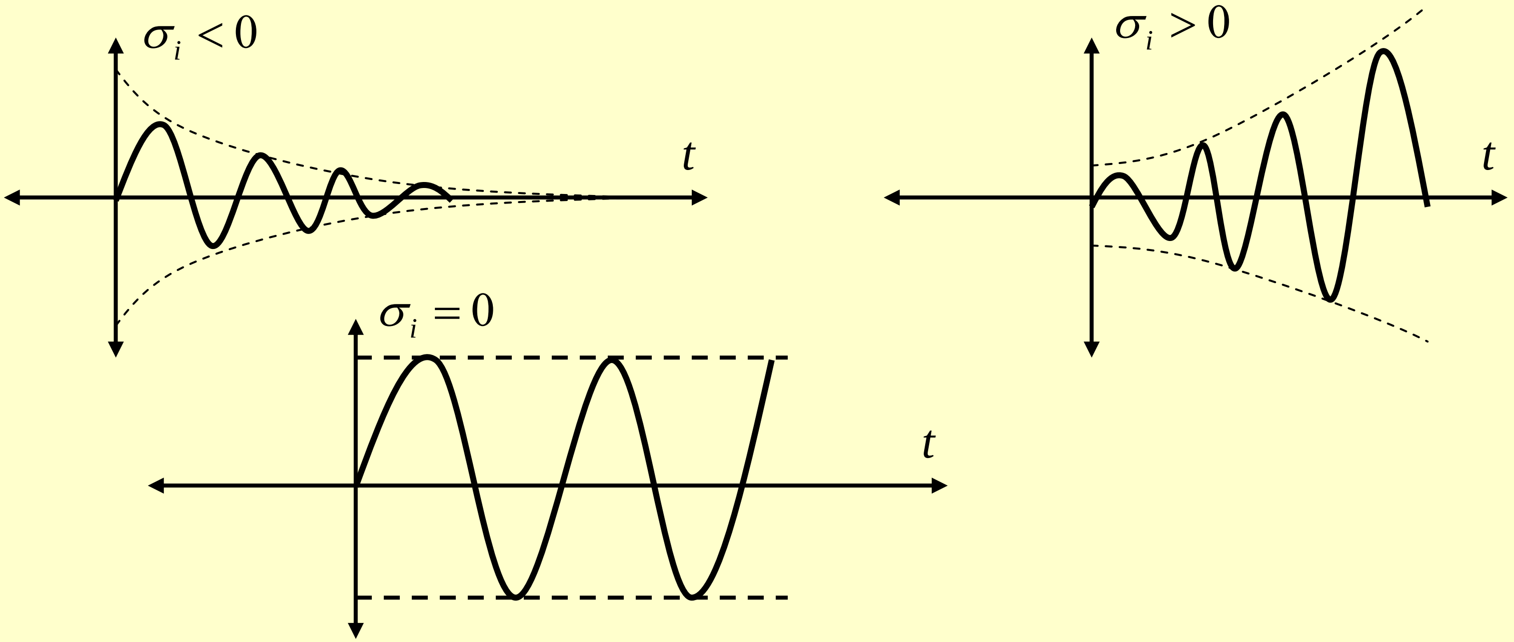
$\left\{e^{\lambda_i t}\right\}_{i=1}^{n}$ ← Set of characteristic modes

Real Root: $\lambda_i = \underset{\substack{\uparrow \\ \text{real}}}{\sigma_i} + \underset{\substack{\uparrow \\ \text{real}}}{j0} \quad \Rightarrow \quad e^{\sigma_i t}$



Complex Root: $\lambda_i = \sigma_i + j\omega_i$

Mode: $e^{\lambda_i t} = e^{\sigma_i t} + e^{j\omega_i t}$



To get only real-valued solutions requires the system coefficients to be real-valued.

⇒ Complex roots of C.E. will appear in conjugate pairs:

$$\left.\begin{aligned} \lambda_i &= \sigma + j\omega \\ \lambda_k &= \sigma - j\omega \end{aligned}\right\}$$ Conjugate pair

$$c_i e^{\lambda_i t} + c_k e^{\lambda_k t} = \underset{\frac{C}{2}e^{j\theta}}{\underset{\uparrow}{c_i}} e^{\sigma t} e^{j\omega t} + \underset{\frac{C}{2}e^{-j\theta}}{\underset{\uparrow}{c_k}} e^{\sigma t} e^{j\omega t}$$

For some real C

Use Euler! $Ce^{\sigma t}\cos(\omega t + \theta) \quad t > 0$

Repeated Roots

Say there are *r* repeated roots

$$Q(\lambda) = (\lambda - \lambda_1)^r \underbrace{(\lambda - \lambda_2)(\lambda - \lambda_3)\ldots(\lambda - \lambda_p)}_{} \qquad p = n - r$$

We “can verify” that: $e^{\lambda_i t}, te^{\lambda_1 t}, t^2 e^{\lambda_1 t}, \ldots, t^{r-1} e^{\lambda_1 t}$ satisfy (▲)

ZI Solution:

$$y_{zi}(t) = \underbrace{\left(c_{11} + c_{12}t + \ldots + c_{1r}t^{r-1}\right)}_{\text{effect of } r\text{-repeated roots}} e^{\lambda_i t} + \text{other modes}:$$

See examples on the next several pages

Differential Equation Examples

Find the zero-input response (i.e., homogeneous solution) for these three Differential Equations.

Example (a)

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{df(t)}{dt}$$

w/ I.C.'s

$$D^2 y(t) + 3Dy(t) + 2y(t) = Df(t)$$

The zero-input form is:

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0$$

$$D^2 y(t) + 3Dy(t) + 2y(t) = 0$$

The Characteristic Equation is:

$$\lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad (\lambda + 1)(\lambda + 2) = 0$$

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The Characteristic Roots are:

$$\lambda_1 = -1 \quad \& \quad \lambda_2 = -2$$

The Characteristic “Modes” are:

$$e^{\lambda_1 t} = e^{-t} \quad \& \quad e^{\lambda_2 t} = e^{-2t}$$

The zero-input solution is:

$$y_{zi}(t) = C_1 e^{-t} + C_2 e^{-2t}$$

The System forces this form through its Char. Eq.

The IC’s determine the specific values of the C_i’s

The zero-input solution is:

$$y_{zi}(t) = C_1 e^{-t} + C_2 e^{-2t}$$

and it must satisfy the ICs so:

$$0 = y_{zi}(0) = C_1 e^{-0} + C_2 e^{-0} \quad \Rightarrow \quad C_1 + C_2 = 0$$

The derivative of the z-s soln. must also satisfy the ICs so:

$$-5 = y'_{zi}(0) = -C_1 e^{-0} - 2C_2 e^{-0} \quad \Rightarrow \quad C_1 + 2C_2 = 5$$

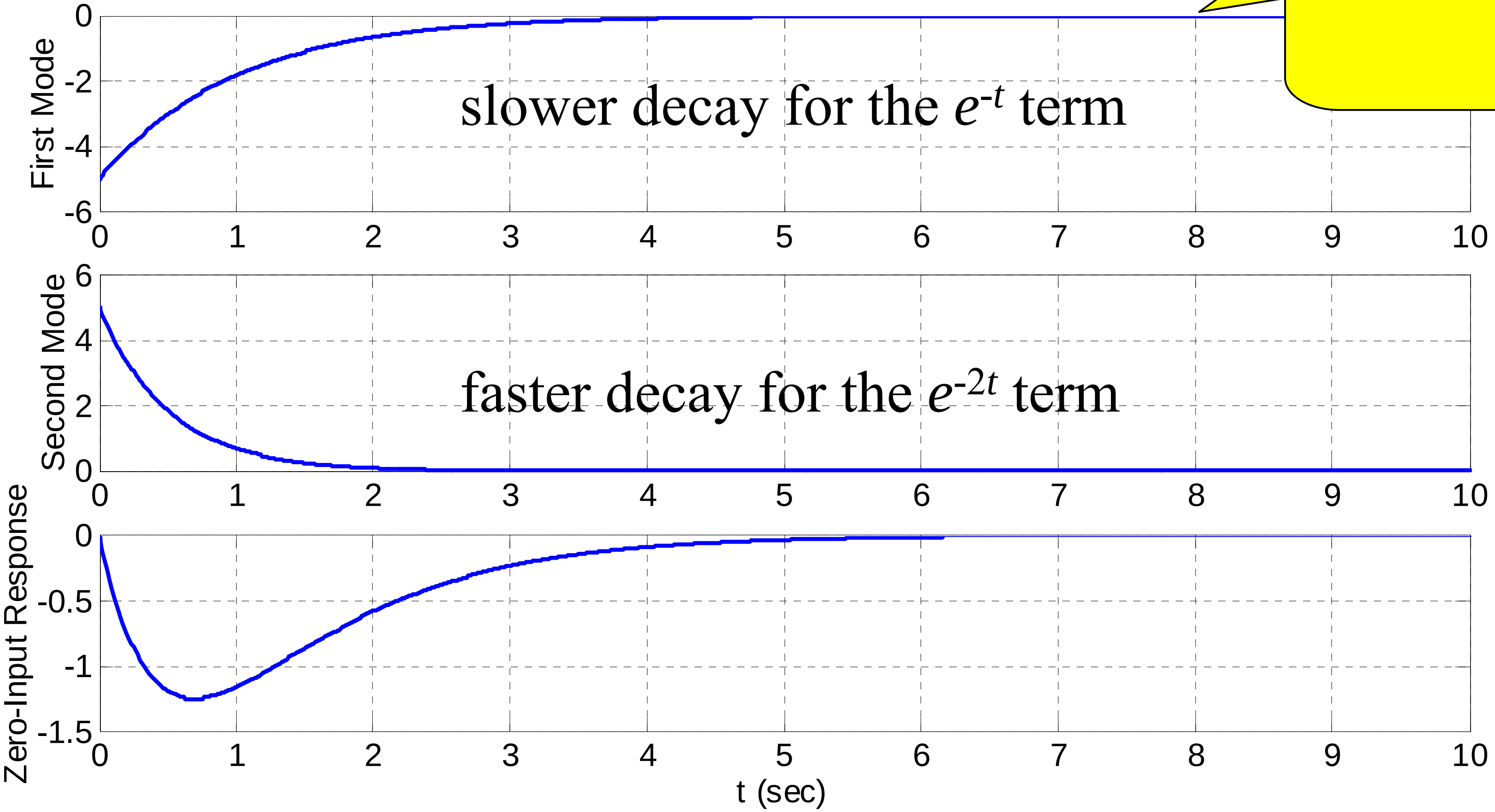
Two Equations in Two Unknowns leads to:

$$C_1 = -5 \quad \& \quad C_2 = 5$$

The "*particular*" zero-input solution is:

$$y_{zi}(t) = \underbrace{-5e^{-t}}_{\text{first mode}} + \underbrace{5e^{-2t}}_{\text{second mode}}$$

Plots for Example (a)



slower decay for the e^{-t} term

faster decay for the e^{-2t} term

Because the characteristic roots are real and negative…
the modes and the Z-I response all decay to zero w/o oscillations

Example (b):

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 9y(t) = 3\frac{df(t)}{dt} + 5f(t)$$

$$D^2y(t) + 6Dy(t) + 9y(t) = 3Df(t) + 5f(t)$$

w/ I.C.'s

The zero-input form is:

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 9y(t) = 0$$

$$D^2y(t) + 6Dy(t) + 9y(t) = 0$$

The Characteristic Equation is:

$$\lambda^2 + 6\lambda + 9 = 0 \quad \Rightarrow \quad (\lambda + 3)^2 = 0$$

The Characteristic Equation is:

$$\lambda^2 + 6\lambda + 9 = 0 \quad \Rightarrow \quad (\lambda + 3)^2 = 0$$

The Characteristic Roots are:

$$\lambda_1 = -3 \quad \& \quad \lambda_2 = -3$$

The Characteristic "Modes" are:

$$e^{\lambda_1 t} = e^{-3t} \quad \& \quad te^{\lambda_2 t} = te^{-3t}$$

Using the "rule" to handle repeated roots

The zero-input solution is:

$$y_{zi}(t) = C_1 e^{-3t} + C_2 te^{-3t}$$

The System forces this form through its Char. Eq.

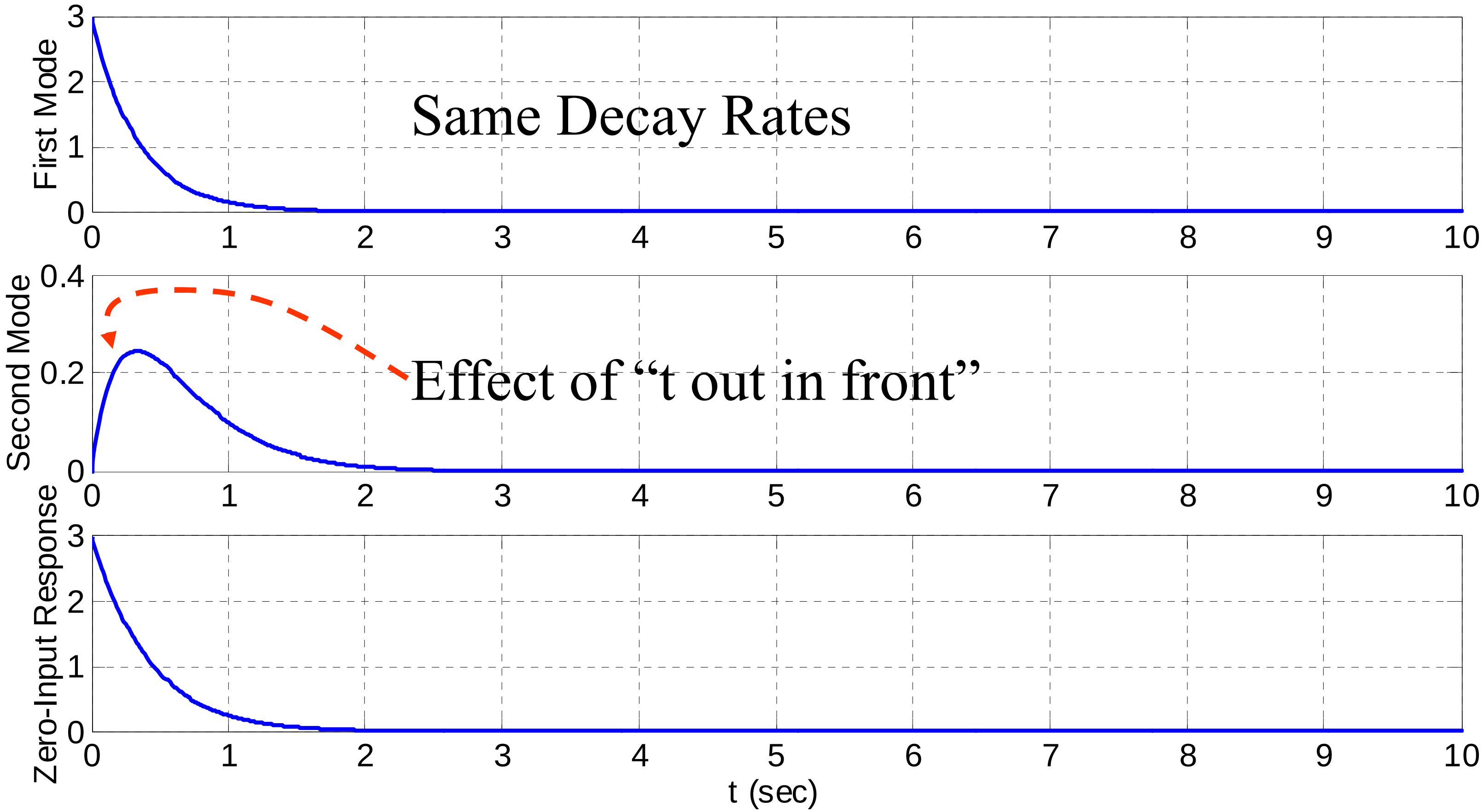
The IC's determine the specific values of the C_i's

Following the same procedure (do it for yourself!!) you get…

The “*particular*” zero-input solution is:

$$y_{zi}(t) = \underbrace{3e^{-3t}}_{\text{first mode}} + \underbrace{2te^{-3t}}_{\text{second mode}} = (3+2t)e^{-3t}$$

Plots for Example (b)



Because the characteristic roots are real and negative…
the modes and the Z-I response all ________________________.

Example (c):

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+40y(t)=\frac{df(t)}{dt}+2f(t)$$

$$D^2y(t)+4Dy(t)+40y(t)=Df(t)+2f(t)$$

w/ I.C.'s

The zero-input form is:

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+40y(t)=0$$

$$D^2y(t)+4Dy(t)+40y(t)=0$$

The Characteristic Equation is:

$$\lambda^2+4\lambda+40=0 \quad \Rightarrow \quad (\lambda+2-j6)(\lambda+2+j6)=0$$

The Characteristic Equation is:

$$\lambda^2 + 4\lambda + 40 = 0 \quad \Rightarrow \quad (\lambda + 2 - j6)(\lambda + 2 + j6) = 0$$

The Characteristic Roots are:

$$\lambda_1 = -2 + j6 \quad \& \quad \lambda_2 = -2 - j6$$

The Characteristic "Modes" are:

$$e^{\lambda_1 t} = e^{-2t}e^{+j6t} \quad \& \quad e^{\lambda_2 t} = e^{-2t}e^{-j6t}$$

The zero-input solution is:

$$y_{zi}(t) = C_1 e^{-2t}e^{+j6t} + C_2 e^{-2t}e^{-j6t}$$

The System forces this form through its Char. Eq.

The IC's determine the specific values of the C_i's

Following the same procedure with some manipulation of complex exponentials into a cosine…

The "*particular*" zero-input solution is:

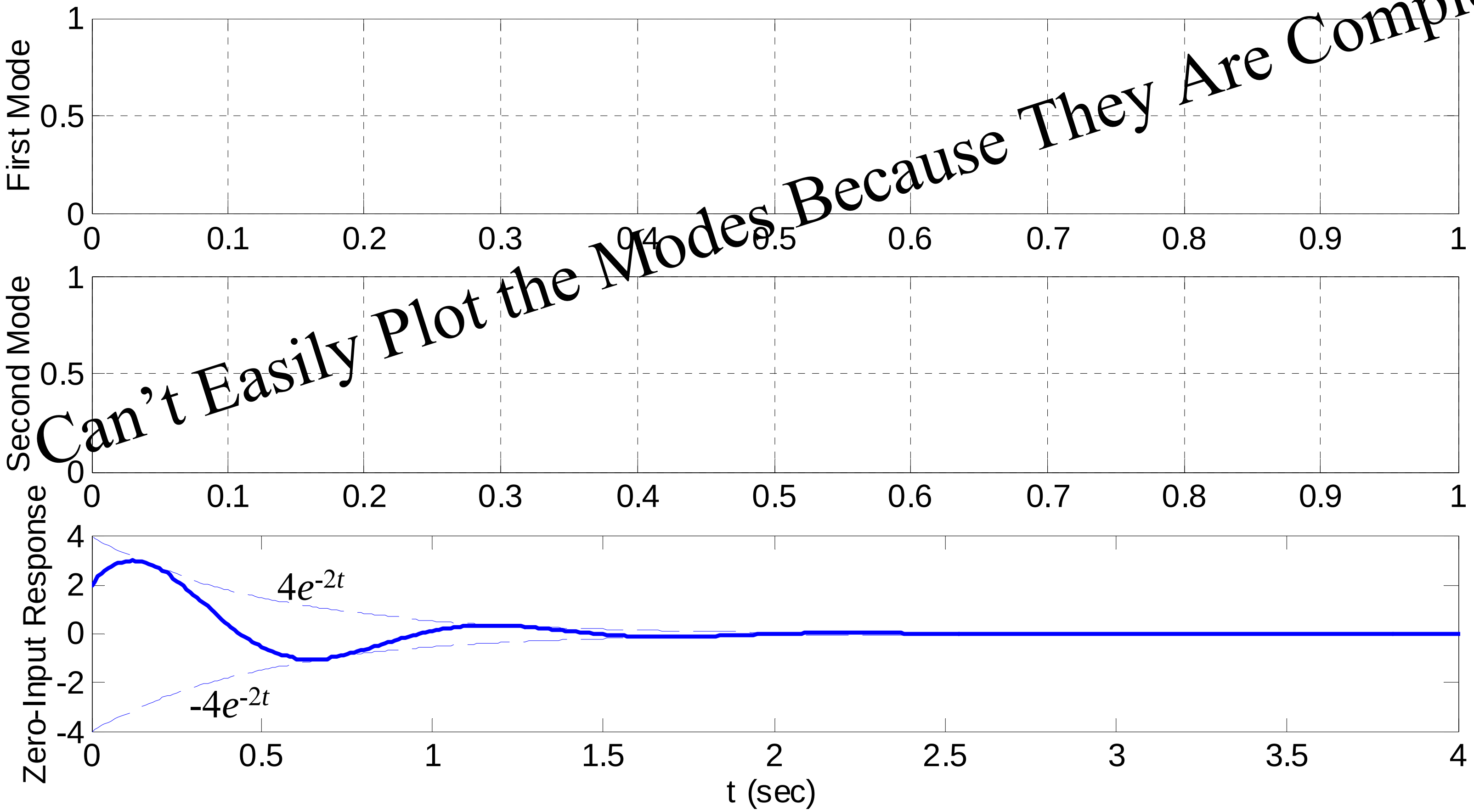
Set by the ICs

$$y_{zi}(t) = 4e^{-2t}\cos(6t + \pi/3)$$

Imag. part of root controls oscillation

Real part of root controls Decay

Plots for Example (c)



Can’t Easily Plot the Modes Because They Are Complex

Because the characteristic roots are complex… have oscillations!

Because real part of root is negative… ___________________!!!

Big Picture…

The structure of the D.E. determines the char. roots, which determine the "character" of the response:

- Decaying vs. Exploding (controlled by real part of root)
- Oscillating or Not (controlled by imag part of root)

The D.E. structure is determined by the physical system's structure and component values.