Binghamton University
State University of New York

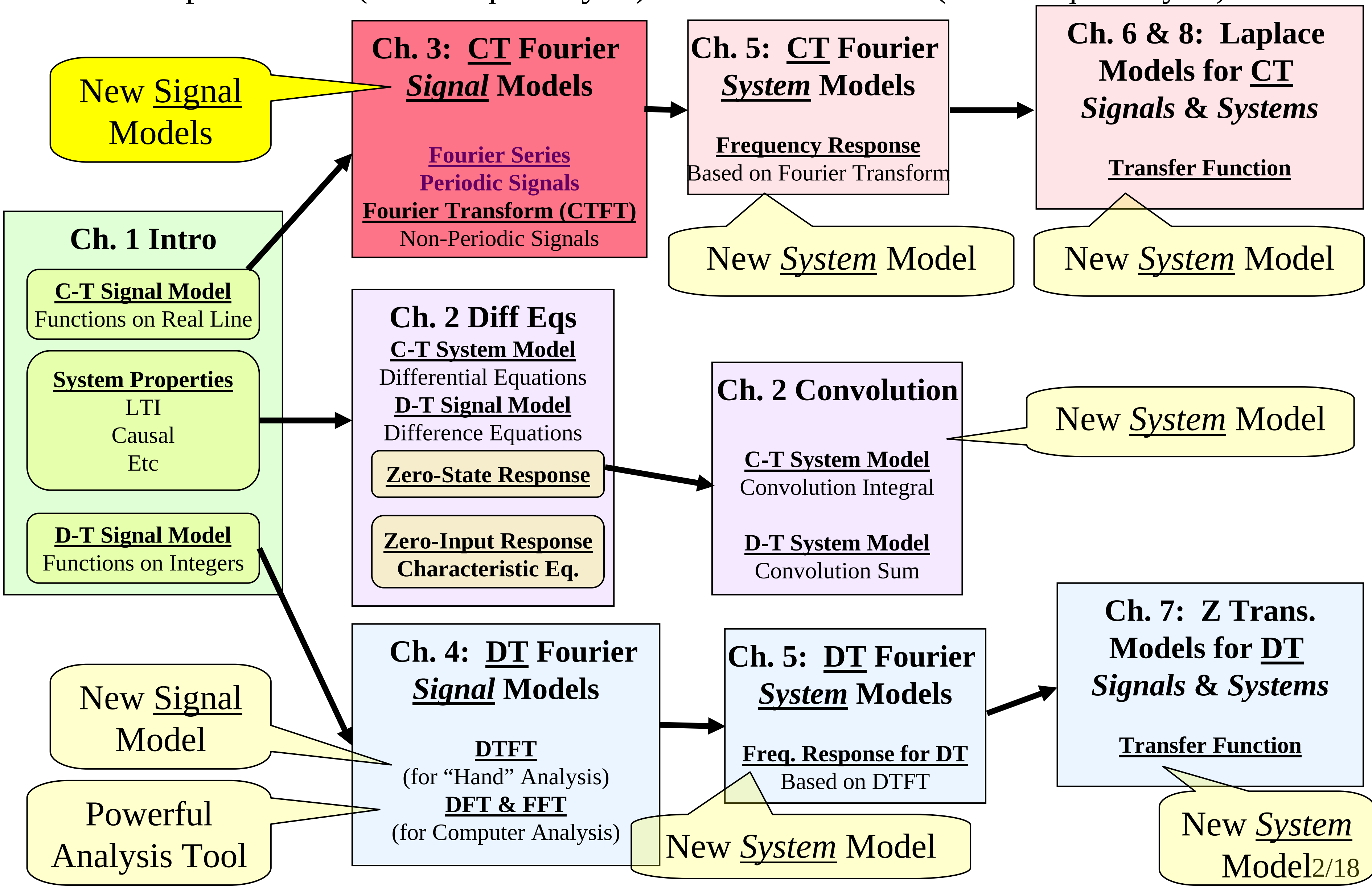
EECE 301
Signals & Systems
Prof. Mark Fowler

<u>Note Set #12</u>

- C-T Signals: Motivation for Fourier Series
- Reading Assignment: Section 3.1 of Kamen and Heck

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



Ch. 3: Fourier Series & Fourier Transform

(This chapter is for C-T case only)

i.e., sinusoids

3.1 Representation in terms of frequency components

- Section 3.1 motivates the following VERY important idea:

  "signals can be built from **sinusoids**"

- Then Sections 3.2 – 3.3 take this idea further to precisely answer

  How can we use **sinusoids** to build periodic signals?

- Then Section 3.4 – 3.7 take this idea even further to precisely answer

  How can we use **sinusoids** to build more-general non-periodic signals?

LTI System

Q: Why all this attention to **sinusoids**?
A: Recall "sinusoidal analysis" in RLC circuits:
Fundamental Result: Sinusoid In $\Rightarrow$ Sinusoid Out

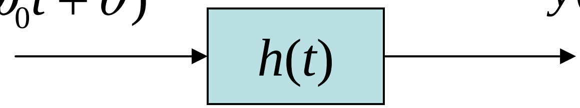
And… it is easy to find out how sinusoids go through an LTI system

“Why Study Response to Sinusoids” (not in Ch. 3… See 5.1)

LTI: **Linear**, Time-Invariant

Q: How does a sinusoid go through an LTI System?

Consider: $x(t) = A\cos(\omega_0 t + \theta)$ $\qquad$ $y(t) = ?$



To make this easier to answer (yes… this makes it easier!!) we use Euler’s Formula:

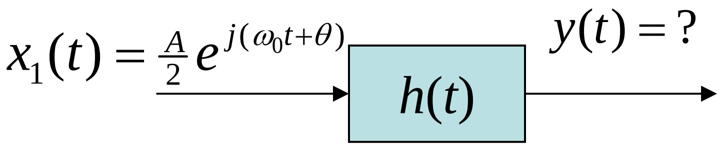
$$x(t) = A\cos(\omega_0 t + \theta) = \underbrace{\frac{A}{2} e^{j(\omega_0 t + \theta)}} + \underbrace{\frac{A}{2} e^{-j(\omega_0 t + \theta)}}$$

The input is now viewed as the sum of two parts… By linearity of the system we can find the response to each part and then add them together.

So we now re-form our question…

Q: How does a *complex* sinusoid go through an LTI System?

Consider:



With convolution as a tool we can now easily answer this question:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau$$

Plug in our input for $x(t\text{-}\tau)$

$$= \int_{-\infty}^{\infty} \frac{A}{2} e^{j[\omega_o (t-\tau) + \theta]} h(\tau) d\tau = \int_{-\infty}^{\infty} \frac{A}{2} e^{j[\omega_o t + \theta]} e^{-j\omega_o \tau} h(\tau) d\tau$$

Use rules for exponentials

$$= \underbrace{\frac{A}{2} e^{j[\omega_o t + \theta]}}_{} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega_o \tau} d\tau}_{\triangleq H(\omega_o)}$$

Pull out part that does not depend on variable of integration… Note that it is just $x_1(t)$

Evaluates to some complex number that depends on $h(t)$ and ω_o

So… the output is just this complex sinusoidal input multiplied by some complex number!!!

So… $y(t) = H(\omega_o)\frac{A}{2}e^{j(\omega_o t+\theta)}$

Complex-valued

Let's work this equation a bit more to get a more useful, but equivalent form…

Because it is complex we can write $H(\omega_o) = |H(\omega_o)|e^{j\angle H(\omega_o)}$

So using this gives:

$$y(t) = \left(|H(\omega_o)|e^{j\angle H(\omega_o)}\right)\frac{A}{2}e^{j(\omega_o t+\theta)}$$

$$= \left(|H(\omega_o)|\frac{A}{2}\right)e^{j(\omega_o t+\theta+\angle H(\omega_o))}$$

$$y(t) = \underbrace{|H(\omega_o)|\frac{A}{2}}_{\text{System changes the amplitude}}e^{j(\omega_o t+\overbrace{\theta+\angle H(\omega_o)}^{\text{System changes the phase}})}$$

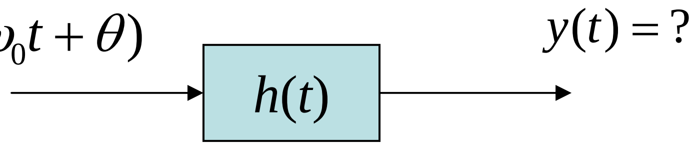
System changes the amplitude

System changes the phase

Now… we can re-visit our first question…

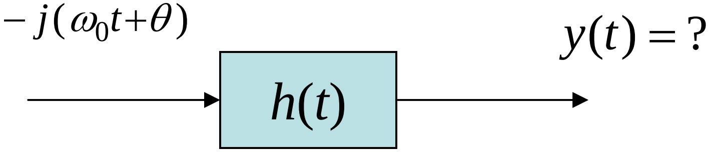
Q: How does a sinusoid go through an LTI System?

Consider: $x(t) = A\cos(\omega_0 t + \theta)$



This is equivalent to:

$$x(t) = \tfrac{A}{2} e^{j(\omega_0 t+\theta)} + \tfrac{A}{2} e^{-j(\omega_0 t+\theta)}$$



And due to linearity and the previous result used twice we have:

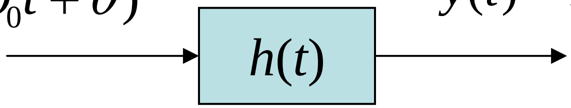
$$y(t) = |H(\omega_o)| \tfrac{A}{2} e^{j(\omega_o t+\theta+\angle H(\omega_o))} + |H(-\omega_o)| \tfrac{A}{2} e^{j(-\omega_o t-\theta+\angle H(-\omega_o))}$$

Later we'll see that $|H(\omega_o)| = |H(-\omega_o)|$ $\qquad \angle H(-\omega_o) = -\angle H(\omega_o)$

So we get: $$y(t) = |H(\omega_o)| A \underbrace{\left[\tfrac{1}{2} e^{j(\omega_o t+\theta+\angle H(\omega_o))} + \tfrac{1}{2} e^{-j(\omega_o t+\theta+\angle H(\omega_o))}\right]}_{\cos(\omega_o t+\theta+\angle H(\omega_o))}$$

So… How does a sinusoid go through an LTI System?

Consider: $x(t) = A\cos(\omega_0 t + \theta)$



$y(t) = A|H(\omega_o)|\cos(\omega_o t + \theta + \angle H(\omega_o))$

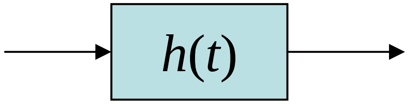
The only thing an LTI system does to a sinusoid is change its amplitude and its phase!!!!

But what about when we have more complicated input signals???

We've already seen that we have to do convolution to solve that case!!!

But… if we have a signal that is a sum of sinusoids then we could use this easy result because of linearity and superposition!!!

$$x(t) = A_1\cos(\omega_1 t + \theta_1) + A_2\cos(\omega_2 t + \theta_2)$$



$$y(t) = A_1|H(\omega_1)|\cos(\omega_1 t + \theta_1 + \angle H(\omega_1)) + A_2|H(\omega_2)|\cos(\omega_2 t + \theta_2 + \angle H(\omega_2))$$

So… breaking a signal into sinusoidal parts makes our job EASY!!
(As long we know what the *H*(ω) function looks like… that is Ch. 5's Problem)

What we look at in Ch. 3 is…

What kind of signals can we use this trick on?

Or in other words… What kinds of signals can we build by adding together sinusoids??!!!

Now…back to Ch. 3!

Let ω_0 be some given "fundamental" frequency

Q: What can I build from building blocks that looks like:

$$A_k \cos(\underbrace{k\omega_0} + \theta_k) \ \ ?$$

Only frequencies that are *integer* multiples of ω_0

Ex.: ω_0 = 30 rad/sec then consider 0, 30 60, 90, …

Ex. 3.1 (Motivation to answer this question!)

My notation is a bit different than the book

A little experiment:

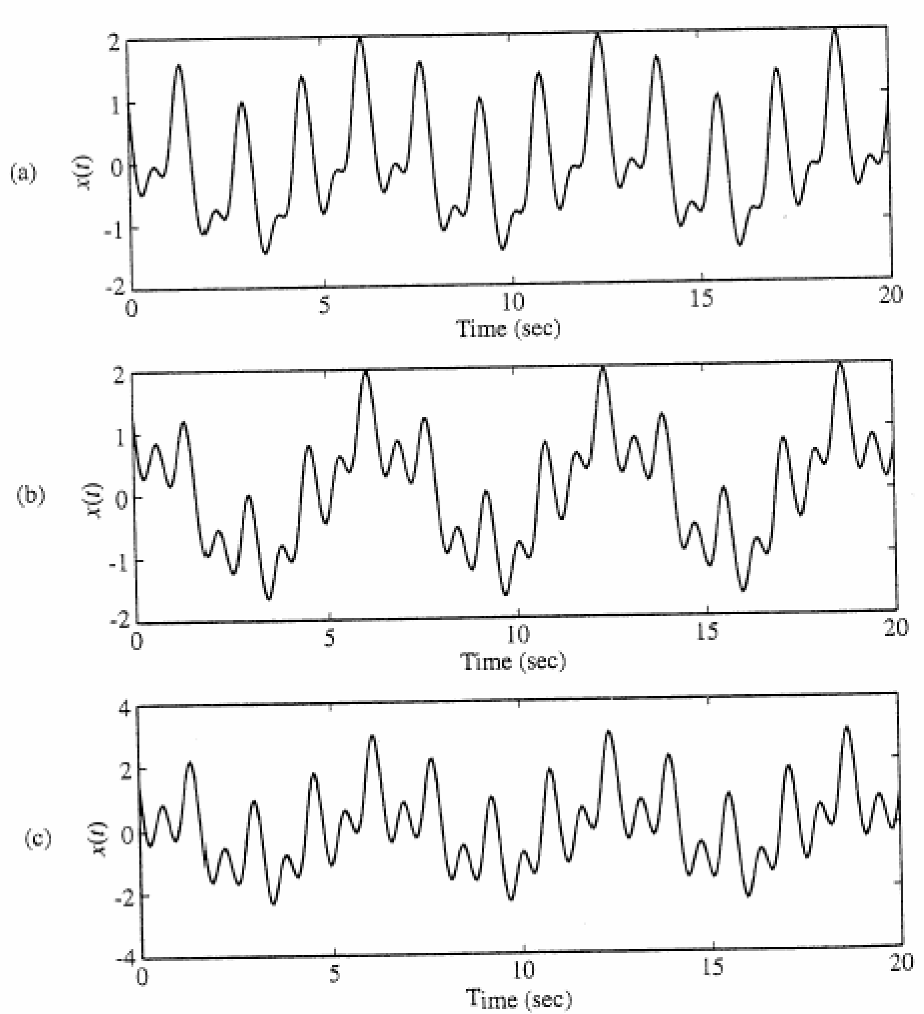
$$x(t) = A_1 \cos(t) + A_4 \cos(4t + \frac{\pi}{3}) + A_8 \cos(8t + \frac{\pi}{2})$$

$\omega_0 = 1$ $\quad 4\omega_0$ $\quad 8\omega_0$

$A_1 = 0.5$
$A_4 = 1$
$A_8 = 0.5$

$A_1 = 1$
$A_4 = 0.5$
$A_8 = 0.5$

$A_1 = 1$
$A_4 = 1$
$A_8 = 1$



Q: How can we easily convey the information about this signal model?

A: Give a plot that shows the amplitude and phase at each frequency!

Amplitude Spectra

Phase Spectra

All three cases are the same

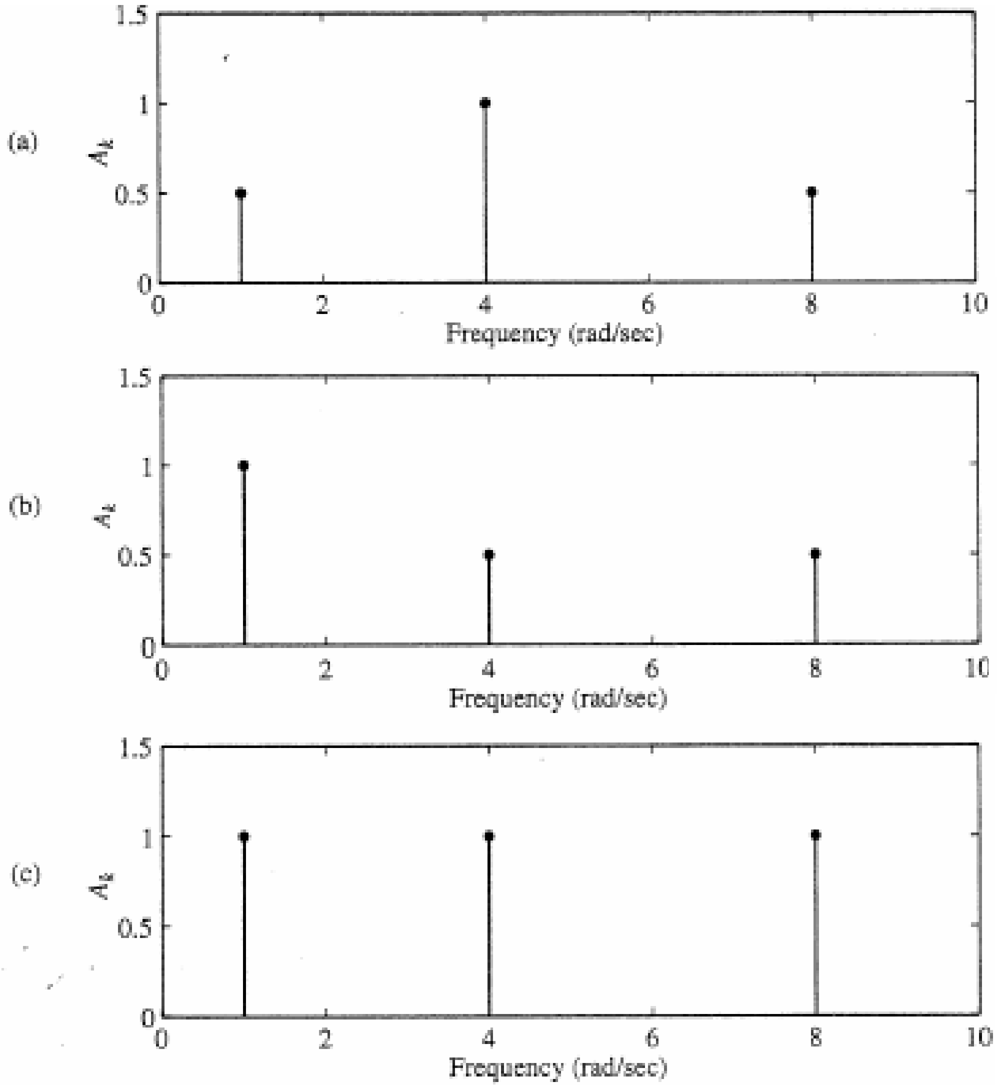
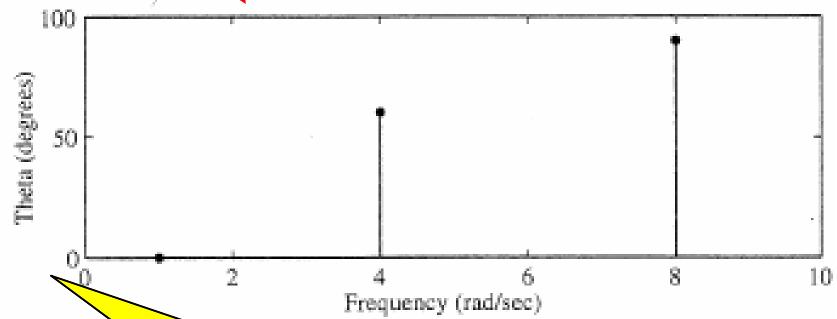


Figure 4.3 Amplitude spectra of the versions of $x(t)$ plotted in Figure 4.2.



Book uses degrees although it is more correct to plot radians…
Radians is what you must use in this course

So we can write this sum of sinusoids like this:

$$x(t) = \sum_{k=1}^{N} A_k \cos(k\omega_0 t + \theta_k)$$

(A_k Real)

Like (3.1) except mine is <u>less</u> general… I <u>force</u>: $\omega_k = k\omega_0$

So I've got a given set of "<u>frequency components</u>" and my <u>model</u> consists of <u>setting the amplitudes and phases</u> to "desired" values

What if we also let $k = 0$, then we get: $\underbrace{A_0 \cos(0 + \theta_0)}$

Its frequency is 0 rad/sec (0 Hz)

= constant, so just use A_0 & $\theta_0 = 0$

⇒ It is a "DC term"

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(k\omega_0 t + \theta_k)$$

"DC offset" (A_0 & A_k Real)

This is called "Trigonometric Form" (Later we'll let it have an infinite # of terms)

⇒ Adding a DC Offset term just moves the <u>whole</u> signal up or down

Trigonometric Form makes the most physical sense but mathematically we often prefer the equivalent **"complex exponential form"**

$$x(t) = c_0 + \left(\sum_{k=1}^{N} c_k e^{jk\omega_0 t} \right) + \left(\sum_{k=-1}^{-N} c_k e^{jk\omega_0 t} \right)$$

c_0 is Real

c_k is Complex

Positive freq. terms

Negative freq. terms

$$x(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}$$

This is called "Complex Exponential Form"

Q: How do we get the Complex Exponential Form from the Trigonometric Form?

A: Euler's Formula!

Each Term in the Trigonometric Form gives…

Two Terms in the Complex Exponential Form (Except the A_0 term)

<u>The details on how to get the Complex Exponential Form</u>:

From direct application of Euler's Formula to each term in the Trigonometric Form of the Fourier Series we get:

$$A_k \cos(k\omega_0 t + \theta_k) = \frac{A_k e^{j(k\omega_0 t + \theta_k)} + A_k e^{-j(k\omega_0 t + \theta_k)}}{2}$$

$k = 1, 2, 3, \ldots$

$\omega_0 > 0$

$$= \left[\frac{A_k}{2} e^{j\theta_k}\right] e^{jk\omega_0 t} + \left[\frac{A_k}{2} e^{-j\theta_k}\right] e^{-jk\omega_0 t}$$

$$= \left[\frac{A_k}{2} e^{j\theta_k}\right] e^{jk\omega_0 t} + \left[\frac{A_k}{2} e^{-j\theta_k}\right] e^{jk(-\omega_0)t}$$

<u>Positive</u>-Frequency Term

<u>Negative</u>-Frequency Term

Every <u>physical</u> sinusoid consists of…
one positive-frequency term and one negative-frequency term.

So for this complex exponential form we need a slightly different spectrum plot.

Must show both positive and negative frequencies
Called "Double-Sided Spectrum"

Example: Consider $x(t) = \cos(t) + 0.5\cos(4t + \pi/3) + \cos(8t + \pi/2)$

which is already in Trigonometric Form of the Fourier Series with $\omega_0 = 1$:

$A_1 = 1$ $A_4 = 0.5$ $A_8 = 1$ (all others are 0)

$\theta_1 = 0$ $\theta_4 = \pi/3$ $\theta_8 = \pi/2$

Using the results in the previous slides we can re-write this in Complex Exponential Form of the FS as:

$$x(t) = \left[0.5e^{jt} + 0.5e^{-jt}\right] + \left[0.25e^{j\pi/3}e^{j4t} + 0.25e^{-j\pi/3}e^{-j4t}\right] + \left[0.5e^{j\pi/2}e^{j8t} + 0.5e^{-j\pi/2}e^{-j8t}\right]$$

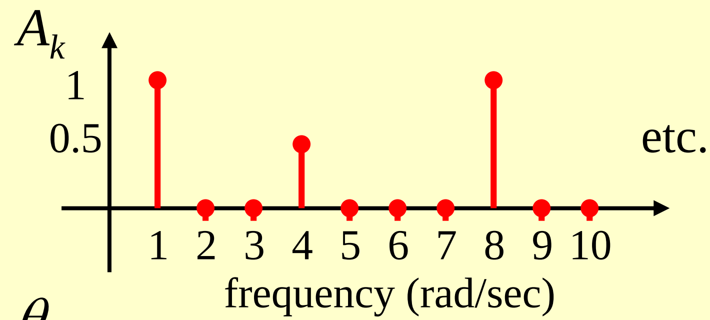
$c_1 = 0.5$ $c_4 = 0.25e^{j\pi/3}$ $c_8 = 0.5e^{j\pi/2}$

$c_{-1} = 0.5$ $c_{-4} = 0.25e^{-j\pi/3}$ $c_{-8} = 0.5e^{-j\pi/2}$ (all others are 0)

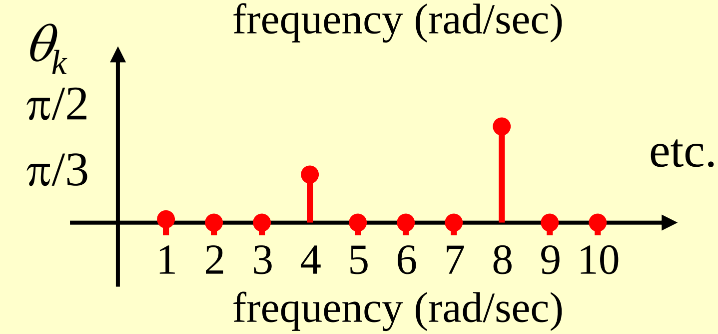
Both Forms Tell Us the Same Information… sometimes one or the other is more convenient

Single-Sided Spectra for Trigonometric Form of FS

Amplitude Spectrum



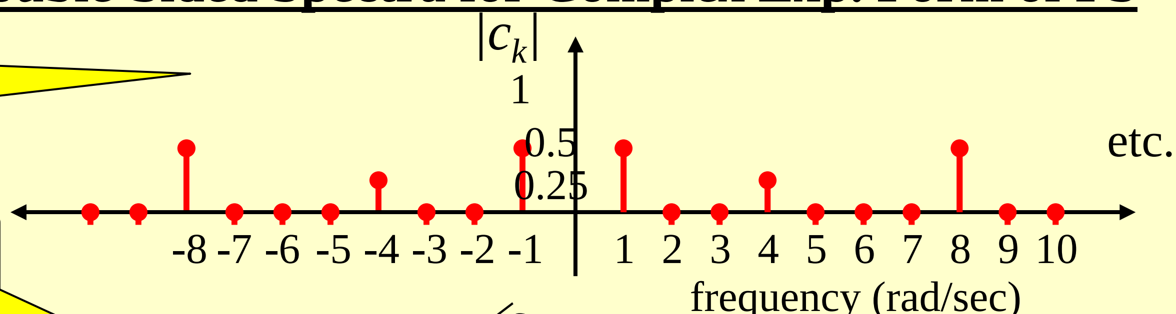
Phase Spectrum



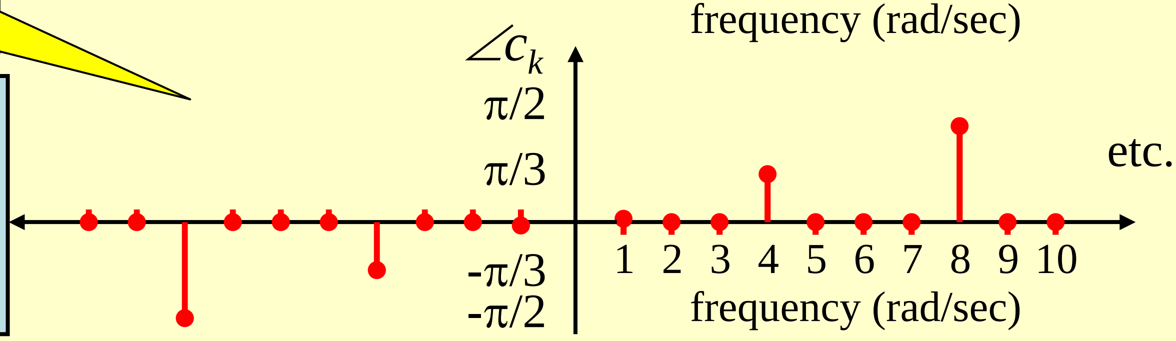
$$x(t) = \cos(t) + 0.5\cos(4t + \pi/3) + \cos(8t + \pi/2)$$

Double-Sided Spectra for Complex Exp. Form of FS

Amplitude Spectrum



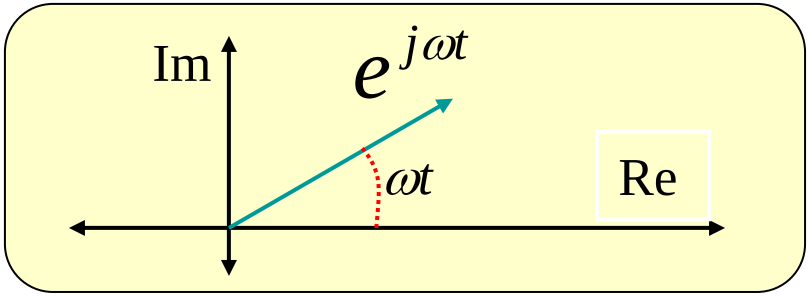
Phase Spectrum



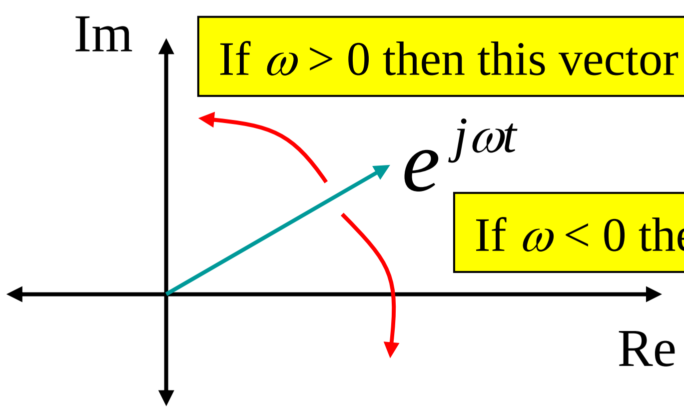
$$x(t) = \left[0.5e^{jt} + 0.5e^{-jt}\right] + \left[0.25e^{j\pi/3}e^{j4t} + 0.25e^{-j\pi/3}e^{-j4t}\right] + \left[0.5e^{j\pi/2}e^{j8t} + 0.5e^{-j\pi/2}e^{-j8t}\right]$$

What do these complex exponential terms look like?

Well… at any fixed time t the function $e^{j\omega t}$ is a complex number with unit amplitude and angle ωt … so we can view it a a vector in the complex plane:



Now… if we let the time variable "flow"… then this vector will rotate:



If $\omega > 0$ then this vector rotates counter-clockwise

If $\omega < 0$ then this vector rotates clockwise

ω controls the angular rate… it has units of rad/sec

Visualizing Rotating Phasors

We know for Euler's Formula that

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

Thus… the real part of a rotating phasor is a cosine wave.

The following Web Demo shows this:

Link to Web Demo for Rotating Phasor

1. Open the web page
2. Click on the box at the top labeled **One**

What you'll see:

1. An Orange phasor rotating around the unit circle
2. The projected Real Part (the "cosine") is shown in Red
3. At the bottom you'll see a vertical axis that represents a time axis and you'll see the Red Real Part repeated and you'll see it tracing out the cosine wave in orange