Binghamton University
State University of New York

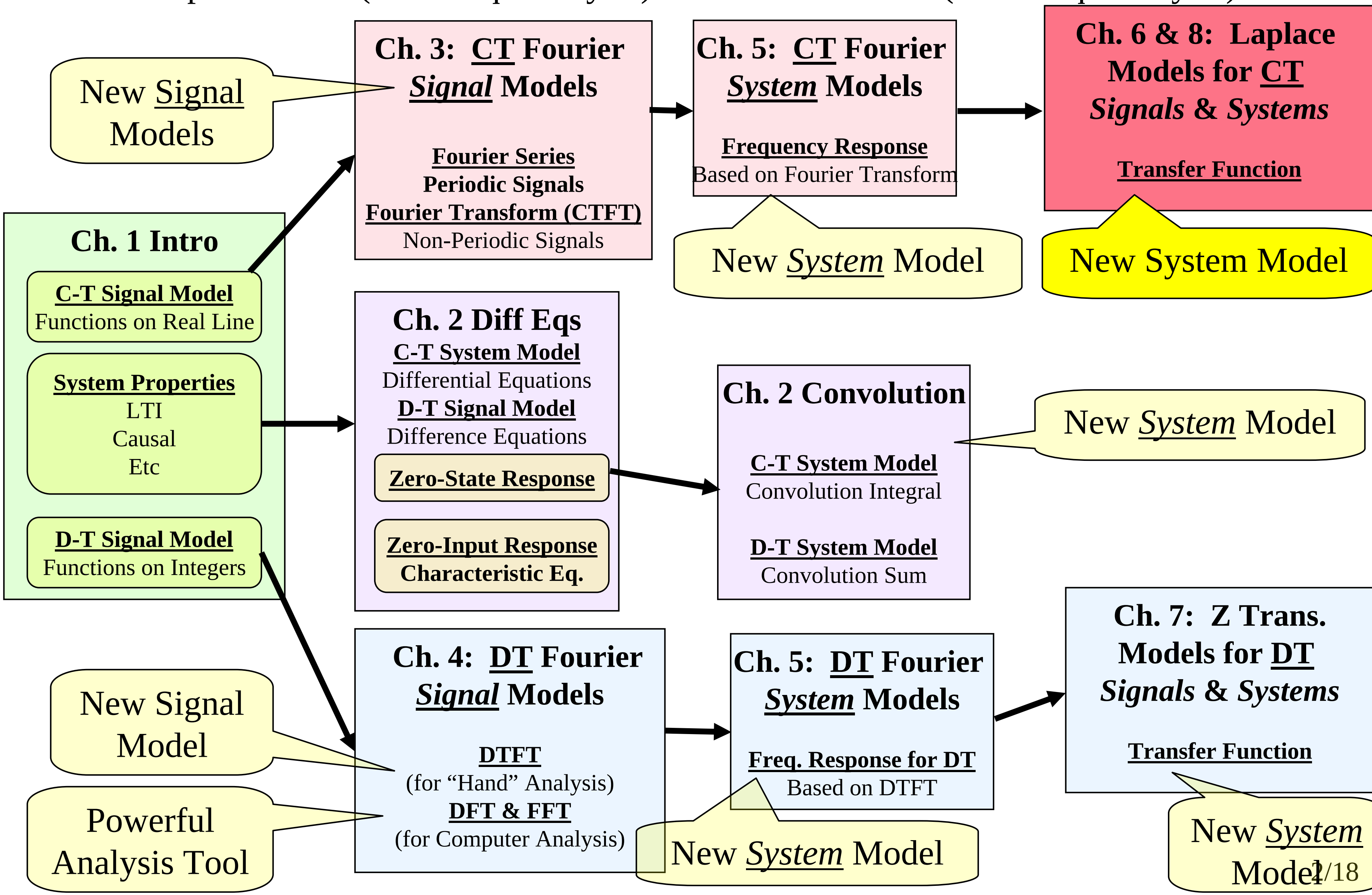
# EECE 301
# Signals & Systems
# Prof. Mark Fowler

## Note Set #27

- C-T Systems: Laplace Transform… “Power Tool” for system analysis
- Reading Assignment: Sections 6.1 – 6.3 of Kamen and Heck

# Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



# <u>What we have seen so far….</u>

- Diff. Equations describe systems
  - Diff<u>erential</u> Eq. for CT
  - Diff<u>erence</u> Eq. for DT
- Convolution with the Impulse Response can be used to analyze the system
  - An integral for CT
  - A summation for DT
- Fourier Transform (and Series) describe what frequencies are in a signal
  - CTFT for CT has an integral form
  - DTFT for DT has a summation form
  - There is a connection between them from the sampling theorem
- The Frequency Response of a system gives a <u>multiplicative</u> method of analysis
  - Freq. Response = CTFT of impulse response for CT system
  - Freq. Response = DTFT of impulse response for DT system

**We now look at two "power tools" for system analysis:**

**Laplace Transform for CT Systems** — Extension of **<u>C</u>**TFT

**Z Transform for DT Systems** — Extension of **<u>D</u>**TFT

# <u>Ch. 6 Laplace Transform & Transfer Function</u>

**Back to <u>C-T</u> signals and systems…**

We've seen that the FT is a useful tool for

-signal analysis (understanding signal structure)

-<u>systems analysis/design</u>

But only if:

Called
"Absolutely Integrable"

1. System is in zero state
2. Impulse response satisfies $\int_{-\infty}^{\infty} |h(t)| dt < \infty$
3. Input satisfies $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

Well… there *<u>are</u>* a few signals that we *<u>can</u>* handle with FT that *<u>do not</u>* satisfy this:
Sinusoids and unit step are two of them

So…frequency response is a tool that can only be used under these three conditions!

The Laplace Transform is a generalization of the CTFT…
it *<u>can</u>* handle cases when these three conditions are *<u>not</u>* met.

There are two analysis methods that the Laplace Transform enables:

<u>Zero state</u> (Sect. 6.5)

LT & Transfer Function

*x*(*t*) and *h*(*t*) may or may not be absolutely integrable

So… this just allows us to do the same thing that the FT does… but for a larger class of signals/systems

<u>Non zero-state</u> (Sect. 6.4)

LT-based solution of differential equations

*x*(*t*) and *h*(*t*) may or may not be absolutely integrable

This not only admits a larger class of signals/systems… it also gives a powerful tool for solving for both the zero-state ***<u>AND</u>*** the zero-input solutions…

***<u>ALL AT ONCE</u>***

<u>First</u> we'll define the LT

<u>Next</u>… See some of its properties

<u>Then</u>… See how to use it in system analysis in these two ways

## Section 6.1: Define the LT

There are 2 types of LT:
- Two-sided (bilateral)
- One-sided (unilateral)

We'll only use the one-sided LT

### Two-Sided LT

$$X_2(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \qquad \text{with} \quad s = \underbrace{\sigma + j\omega}_{\text{complex variable}}$$

The book doesn't do this

### One-Sided LT

$$X_1(s) = \int_{0}^{\infty} x(t)e^{-st}dt \qquad \text{with} \quad s = \overbrace{\sigma + j\omega}^{\text{complex variable}}$$

One-sided LT defined this way → even if $x(t) \neq 0$, $t < 0$

But we will mostly focus on causal systems and causal inputs

One place the LT is most useful is when

Causal signal

1. The system has Initial conditions at $t = 0$

2. Input $x(t)$ is "applied at $t = 0$" $\Rightarrow$ $\underline{x(t) = 0 \quad t < 0}$

(This will be our focus in this course)

For this case: $X_1(s) = X_2(s)$ $\Rightarrow$ Just use $X(s)$ notation (drop the "1" subscript)

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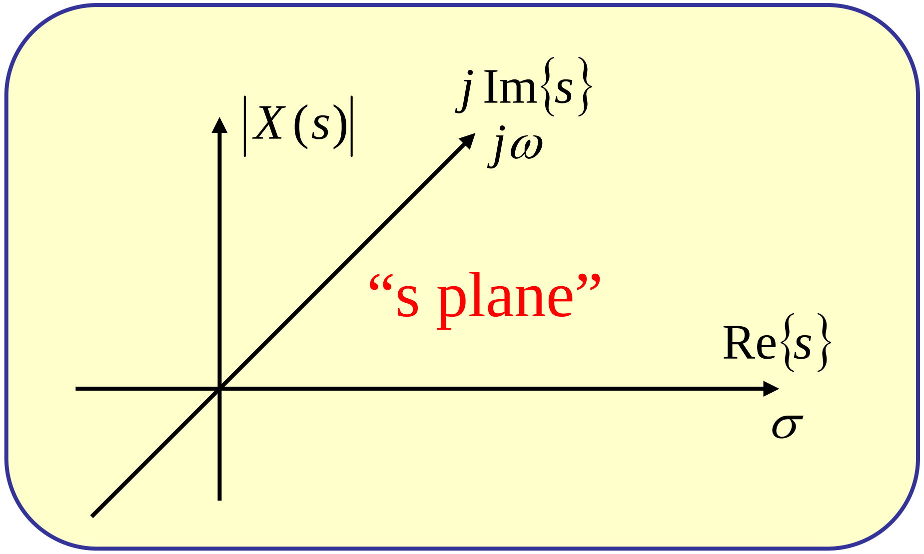
Note that $X(s)$ is: $\begin{cases} \text{a complex valued function} \\ \text{of a complex variable } s = \sigma + j\omega \end{cases}$

Must plot on a plane… the "s-plane"




Similarly for $\angle X(s)$

# Example of Finding a LT

Consider the signal $x(t) = e^{-bt}u(t) \quad b \in \Re$

> Back when we studied the FT we had to limit *b* to being *b* > 0… with the LT we don't need to restrict that!!!

This is a causal signal.

By definition of the LT:

$$X(s) = \int_0^\infty e^{-bt}e^{-st}dt = \int_0^\infty e^{-(s+b)t}dt$$

> This is an easy integral to do!!

> The limit is here by the definition of the integral

$$X(s) = \frac{-1}{s+b}\left[e^{-(s+b)t}\right]_{t=0}^{t=\infty} = \frac{-1}{s+b}\left[\underbrace{\lim_{t\to\infty} e^{-(s+b)t}}_{\text{look at this}} - 1\right]$$

**look at this**

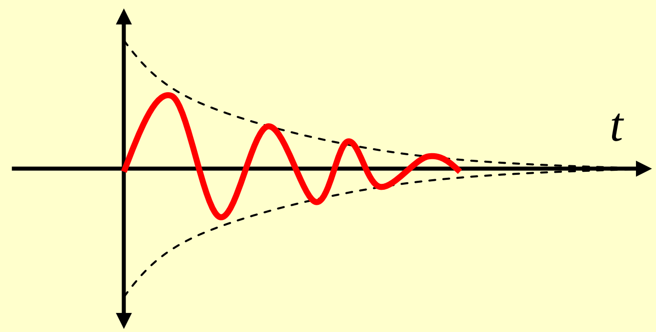
If this limit does not converge… then we say that the integral "does not exist"

So… we need to find out under what conditions this integral exists.

So… let's look at the function inside this limit…

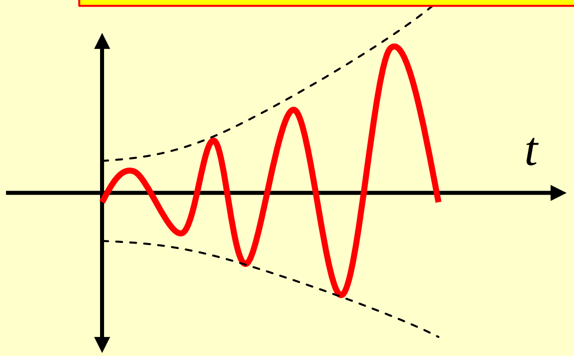
$$e^{-(s+b)t} = e^{-[(\sigma+b)+j\omega]t}$$

if $\sigma + b > 0 \Rightarrow \sigma > -b$




**Has Two Main Behaviors**

if $\sigma + b < 0 \Rightarrow \sigma < -b$




Thus, $\lim_{t\to\infty} e^{-(s+b)t}$ "exists" only for $\sigma > -b$

So, we can't "find" this $X(s)$ for values of $s$ such that Re{$s$} ≤ -$b$

**But for $s$ with Re{$s$} > -$b$ we have no trouble.**

⇒For each $X(s)$ we need to know at which $s$ values "things work"

This set of $s$ is called the "Region of Convergence" (ROC)

Don't worry too much about ROC… at this level it kind of takes care of itself

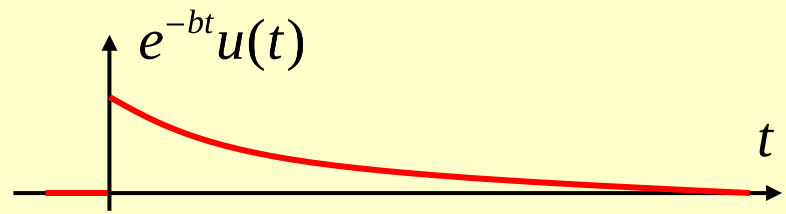
So for $x(t) = e^{-bt}u(t)$ We have $X(s) = \dfrac{1}{s+b} \qquad \text{Re}\{s\} > -b$

This result… and many others… is on the Table of Laplace Transforms that is available on my web site

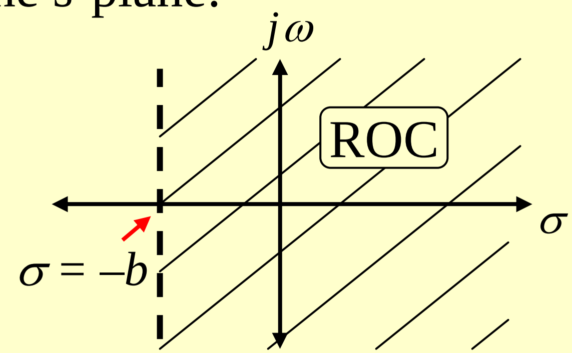
Please use the tables from the website… the ones in the book have some errors on them!!!!

**If $b > 0$** then $x(t)$ itself decays:

For $b > 0$, $-b$ is negative



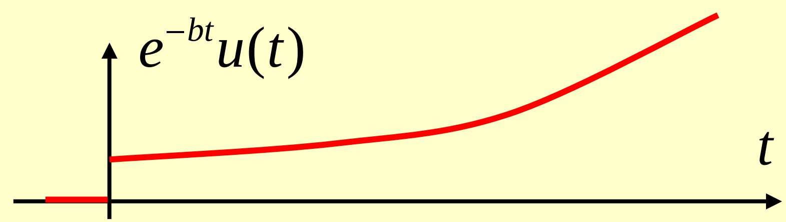

And we have on the s-plane:



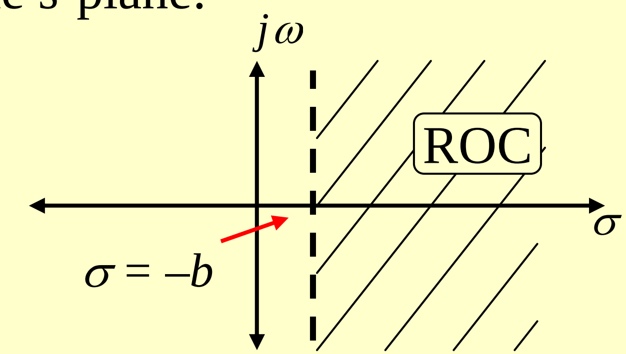

**This case can be handled by the FT… and can also be handled by the LT**

**If $b < 0$** then $x(t)$ itself "explodes":

For $b < 0$, $-b$ is positive




And we have on the s-plane:




**This case can't be handled by the FT… but by restricting our focus to values of $s$ in the ROC, the LT can handle it!!!**

**Connection between FT & LT (for causal signals)**

$$\text{FT}: \; X(\omega) = \int_0^\infty x(t) e^{-j\omega t} dt$$

$$\text{LT}: \; X(\sigma + j\omega) = \int_0^\infty x(t) e^{-(\sigma + j\omega)t} dt$$

**It appears that letting $\sigma = 0$ gives LT = FT…**

**But this is only true if ROC includes the "$j\omega$ axis"!!!**

**If the ROC includes the "$j\omega$ axis"…**

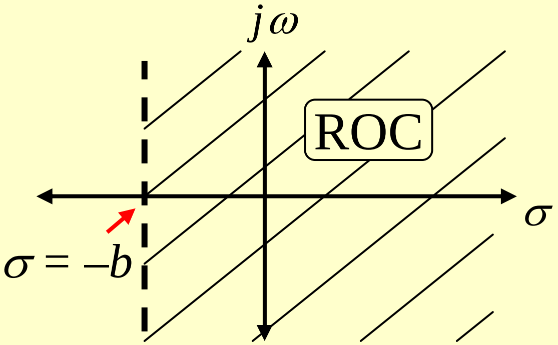
**Then the FT is "embedded" in the LT**

**Get the FT by taking the LT and evaluating it only on the $j\omega$ axis… i.e., take a "slice" of the LT on the $j\omega$ axis**

# Let's Revisit the Example Above

$$x(t) = e^{-bt}u(t) \quad \leftrightarrow \quad X(s) = \frac{1}{s+b} \quad \text{Re}\{s\} > -b$$

If $b > 0$, then ROC includes the "$j\omega$ axis":




$$\Rightarrow X(s)\Big|_{s=j\omega} = \left[\frac{1}{s+b}\right]_{s=j\omega} = \left[\frac{1}{j\omega+b}\right]$$

Same as on FT table

# Section 6.3 Inverse LT

Like the FT…once you know $X(s)$ you can use the inverse LT to get $x(t)$

The definition of the inverse LT is:

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}\,ds$$

with $c$ chosen such that $s = c + j\omega$ is in ROC

This is a "complex line integral" in complex s-plane…

**HARD TO DO!!**

But…if $$X(s) = \frac{b_M s^M + b_{M-1}s^{M-1} + \ldots + b_1 s + b_0}{a_N s^N + a_{N-1}s^{N-1} + \ldots + a_1 s + a_0}$$

Ratio of polynomials in $s$

"Rational Function"

Then its easy to find $x(t)$ using partial fractions and a table of LT pairs

This will be covered in some other notes

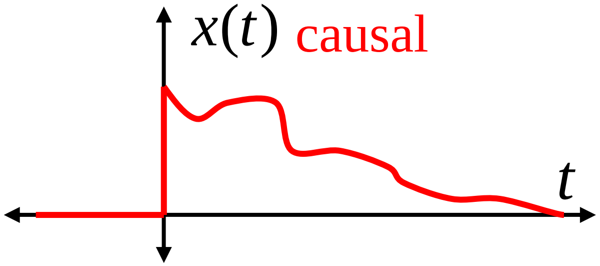
## 6.2 Properties of LT

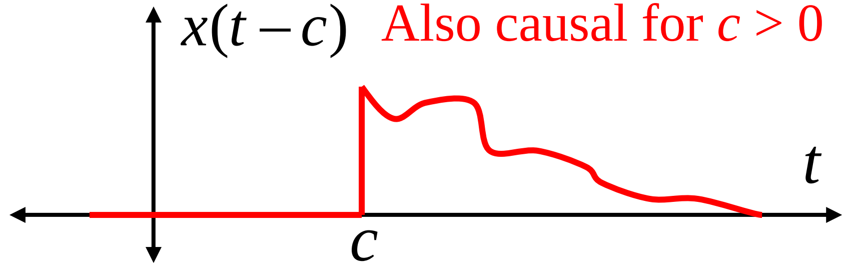
Because of the connection between FT & LT we expect these to be similar to the FT properties we already know!

**Linearity**: $ax(t) + by(t) \quad \leftrightarrow \quad aX(s) + bY(s)$

**Right shift in time (delay):**

Stated here for causal signal (book gives general case)




$$x(t-c) \leftrightarrow e^{-cs} X(s)$$

($c > 0$, $x(t)$ causal)

Compare to time shift for FT: $e^{-j\omega c}$ *vs.* $e^{-cs}$

Recall: $s = \sigma + j\omega$

**Note: There does not exist a result for "left sift" for causal signals and the 1-sided LT**

**Time Scaling:**

Compare to FT

$$x(at) \leftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right)$$

$a > 0$

Note: $a < 0$ makes $x(at)$ non-causal
So we limit to $a > 0$

**Multiply by $t^n$:**

$$t^n x(t) \leftrightarrow (-1)^N \frac{d^N}{ds^N} X(s)$$

**Multiply by Exponential:**

$$e^{at} x(t) \leftrightarrow X(s-a)$$

with $a$ real or complex

Shift in s-plane

**Multiply by sinusoid:**

$$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}\left[X(s+j\omega_0) - X(s-j\omega_0)\right]$$

$$x(t)\cos(\omega_0 t) \leftrightarrow \frac{j}{2}\left[X(s+j\omega_0) + X(s-j\omega_0)\right]$$

Note: Book does not use $\omega_0$ with subscript "*0*"

Warning! So $s - j\omega_0$ above is written $s - j\omega$

Danger! Let $s = \sigma + j\omega \Rightarrow s - j\omega = \sigma$!
Not what is intended!!

**Time Differentiation:**

$$\dot{x}(t) \leftrightarrow sX(s) - x(0)$$

Warning: If $x(t)$ is discontinuous at $t = 0$ then we use $x(0^-)$ instead

Very different from FT property

This LT property allows handling of IC's!!!

**Integration:**

$$\int_0^t x(\lambda)d\lambda \leftrightarrow \frac{1}{s}X(s)$$

These two properties have a nice "opposite" relationship:

**Note**: Differentiation $\Rightarrow$ Multiply by s

Integration $\Rightarrow$ Divide by s

"opposites" "opposites"

These two properties are crucial for linking the LT to the solution of Diff. Eq.

They are also crucial for thinking about "system block diagrams"

**Convolution:** $x(t) * h(t) \quad \leftrightarrow \quad X(s)H(s)$ **Same as for FT!**

Recall: If $h(t)$ is system impulse response

then $H(\omega)$ is system Frequency Response

We'll see that $H(s)$ is system "Transfer Function"

**"Transfer Function"**

is a
generalization of

**"Frequency Response"**

SKIP: **Initial/Final Value Theorems**

These properties… and some others… are on the Table of Laplace Transform Properties that is available on my web site

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