BINGHAMTON UNIVERSITY
State University of New York

EECE 301
Signals & Systems

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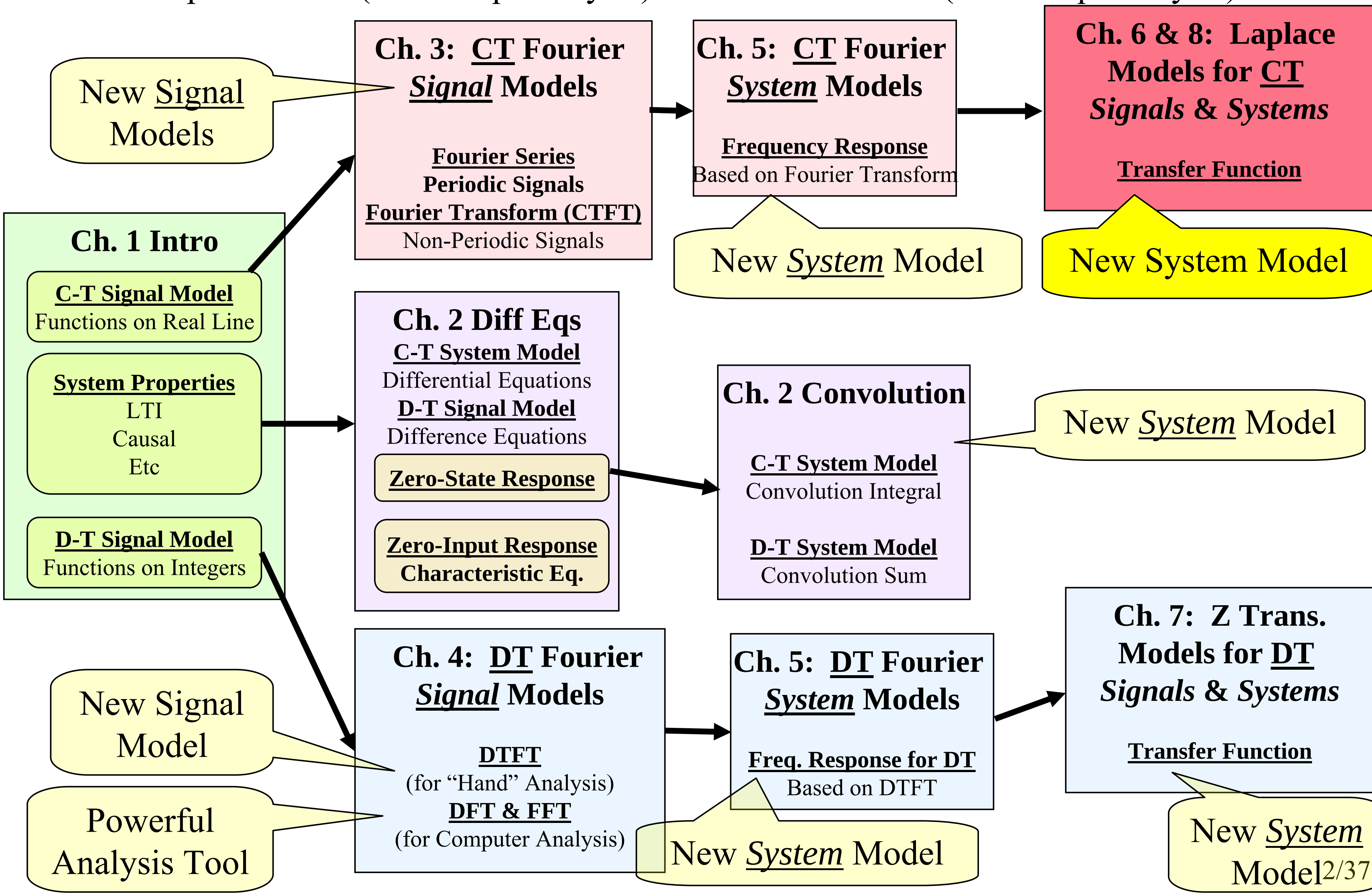
Note Set #32

- C-T Systems: Transfer Function … and Frequency Response
- Reading Assignment: Section 8.5 of Kamen and Heck

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



8.5 Frequency Response Function

We have seen that there are two similar tools that relate the output signal to the input signal:

Transfer Function: $H(s)$

Frequency Response: $H(\omega)$

If the system is stable we know that we can use $H(\omega)$ as a tool…

…and we can easily get $H(\omega)$ from $H(s)$ by replacing $s \rightarrow j\omega$

In analysis/design of systems and circuits it helps to look at plots of:

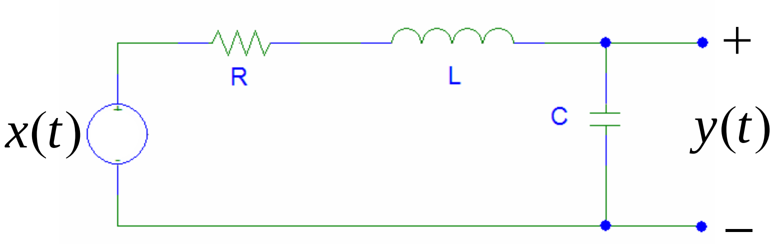
$$\begin{cases} |H(\omega)| & vs. \;\; \omega \\ \angle H(\omega) & vs. \;\; \omega \end{cases}$$

We could just plot over ω > 0, because we know about symmetries

It is, of course, easy to use computers to compute the data and plot it… Anyone can be trained to do that… good engineers are valuable because they *understand* what the plots show!!!!

Example of Computing the Frequency Response

Recall the series RLC circuit…



Recall: $$H(s) = \frac{1/LC}{s^2 + (R/L)s + (1/LC)}$$

Given specific component values: $R = 20\Omega$ $L = 1\text{mH}$ $C = 1\mu\text{F}$

The transfer function then becomes:

$$H(s) = \frac{10^9}{s^2 + (2 \times 10^4)s + 10^9}$$

Now it is possible to replace $s \to j\omega$ and then use general numerical S/W to compute the frequency response….

Or… Use Matlab's "freqs" routine

Next

FREQS Laplace-transform (s-domain) frequency response.

H = freqs(B,A,w) returns the complex frequency response vector H of the filter B/A:

$$H(s)=\frac{B(s)}{A(s)}=\frac{b(1)s^{nb-1}+b(2)s^{nb-2}+\ldots+b(nb)}{a(1)s^{na-1}+a(2)s^{na-2}+\ldots+a(na)}$$

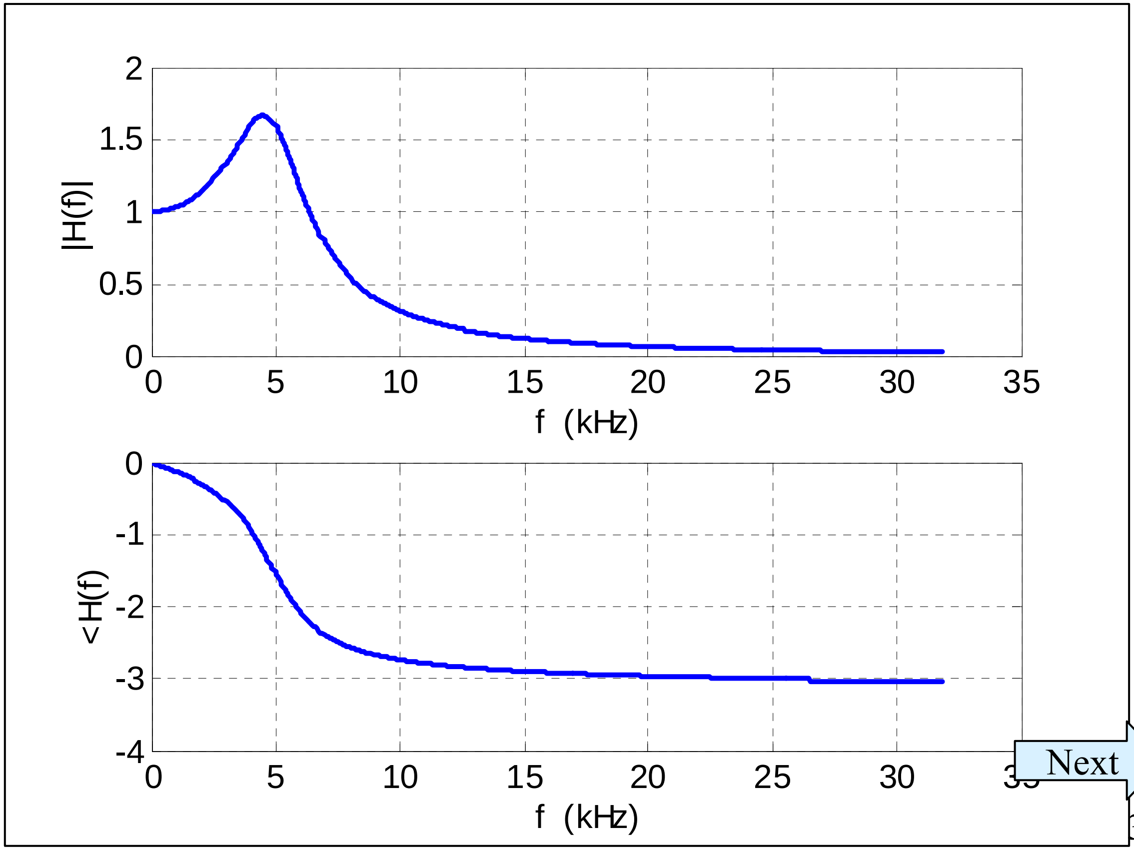
given the numerator and denominator coefficients in vectors B and A.

$$H(s)=\frac{10^9}{s^2+(2\times10^4)s+10^9}$$

```
>> w=0:100:20e4;
>> H=freqs(1e9,[1 2e4 1e9],w);
>> subplot(2,1,1)
```

Create w in units of rad/sec – convert to kHz

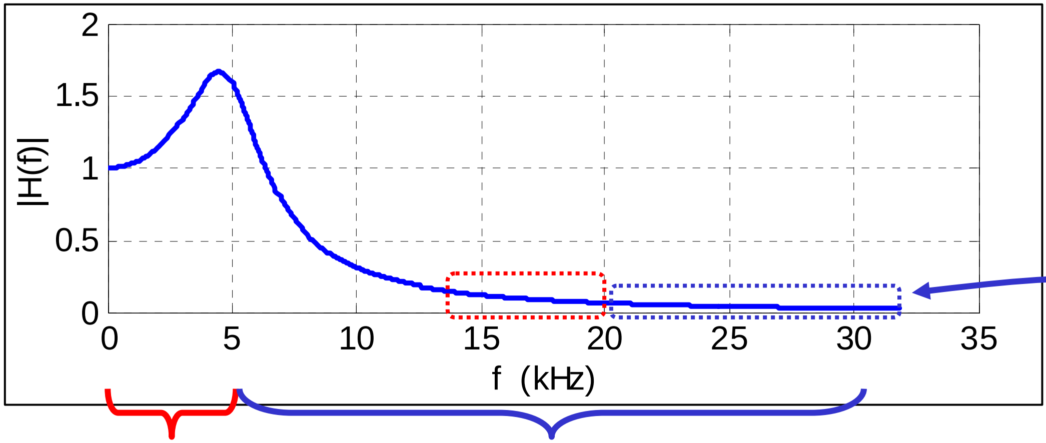
```
>> plot((w/(2*pi))/1e3,abs(H))
>>%%% Plots are vs. f in kHz
>> subplot(2,1,2)
>> plot((w/(2*pi))/1e3,angle(H))
```



Next

Although the previous plots are correct, there are two problems…

Suppose we are interested in using this filter in an audio application:



1. We may be just as interested in 0 – 5 kHz as we are in 5 – 30 kHz
   – But this plot has the 0 – 5 kHz region all "scrunched up"

2. Values of |H| of, say, 0.1 and 0.01 affect the signal by significantly different amounts
   – But they show up looking virtually the same on the plot above

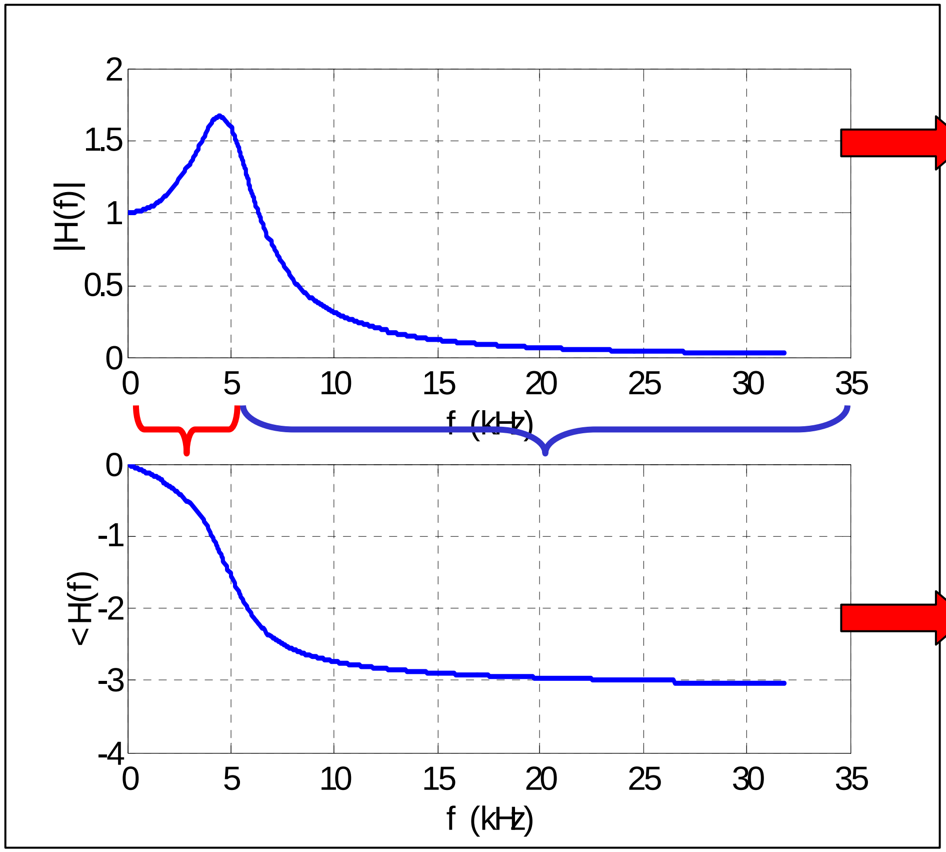
Next

<u>The fixes to these problems are</u>:

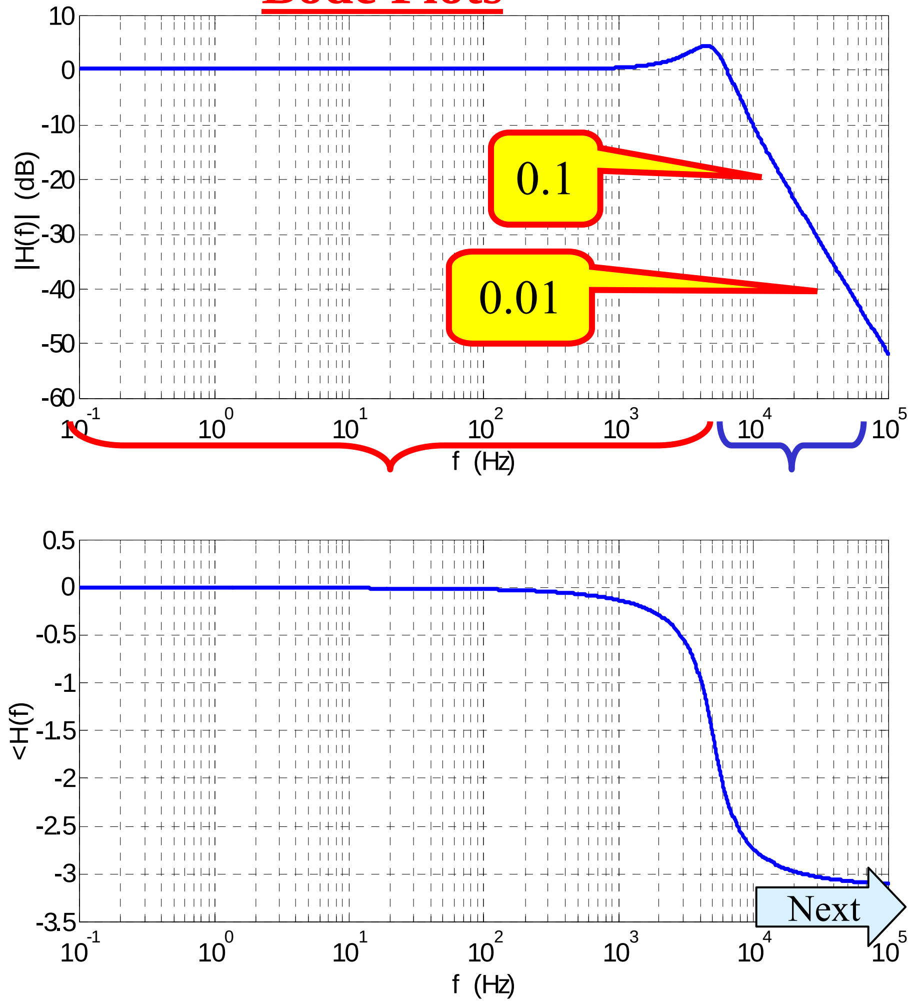
1. **Plot on a logarithmic frequency axis…**
2. **Plot the magnitude in dB…** ← Decibels

Such plots are "<u>Bode Plots</u>"… named after the engineer who introduced them

<u>Our Original Plots</u>



<u>Bode Plots</u>



To convert $|H(\omega)|$ into dB we use $20 \log_{10}(|H(\omega)|)$…

…we'll see more about this in just a bit

Here are the matlab commands to make these plots…

2000 points from 10^{-1} Hz to 10^{5} Hz, equally spaced on a *log* axis

```
f=logspace(-1,5,2000);   % create log-spaced frequencies in Hz
w=2*pi*f;   % convert into rad/sec for use in freqs
H=freqs(1e9,[1 2e4 1e9],w);   % compute H as before
semilogx(f,20*log10(abs(H)))  % plot with log x-axis w/ 20 log10( ) for dB
subplot(2,1,2)
semilogx(f,angle(H))
```

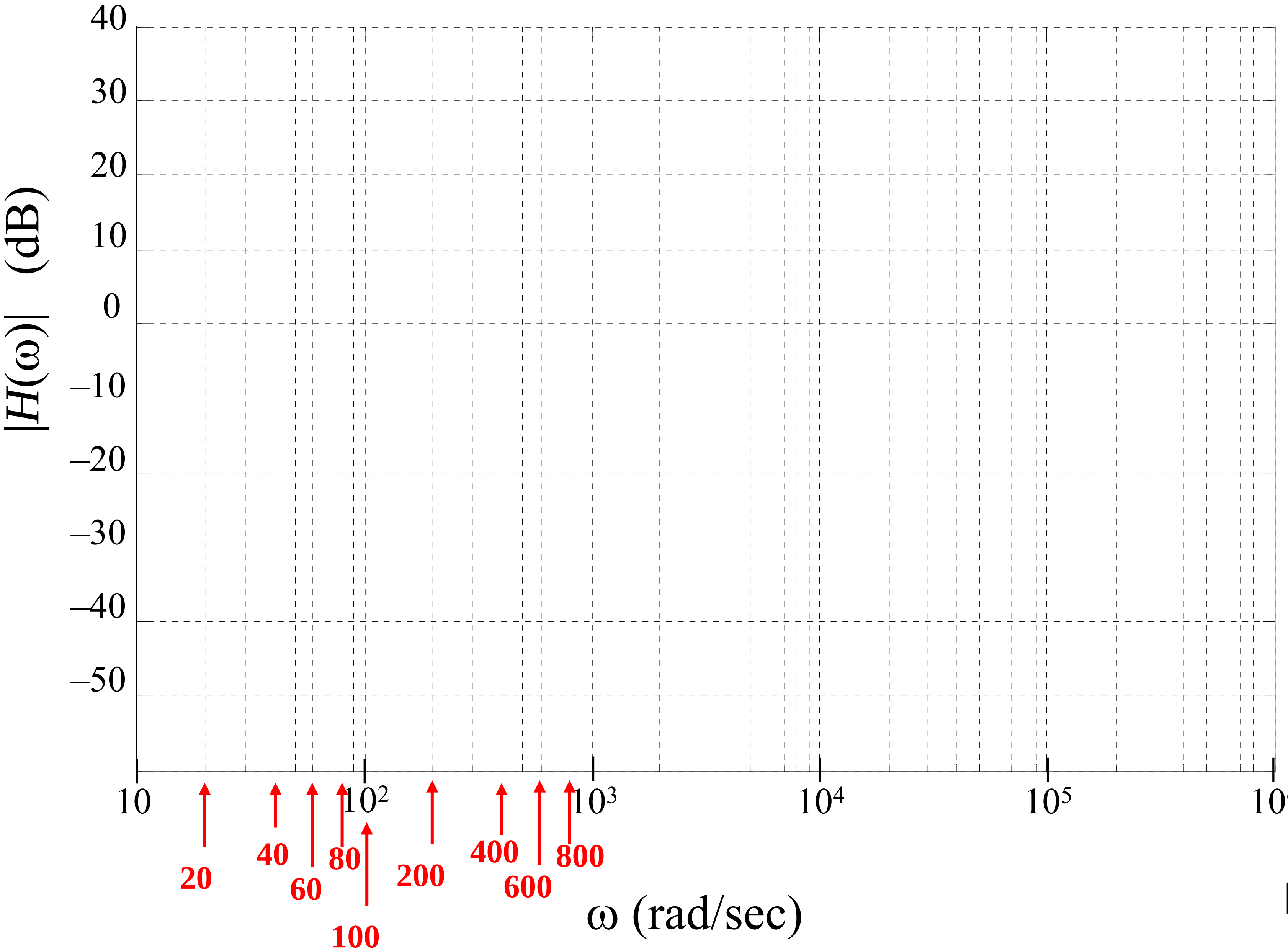
"semilogx" makes a plot with a log x-axis and linear (ordinary) y-axis

Use **20** because |H| is not a power gain (See next few pages)

"semilogx" – gives log x-axis, linear y-axis
"semilogy" – gives linear x-axis, log y-axis
"loglog" – gives log x-axis, log y-axis

Next

A Semi-Log Axis



Next

Defining the Decibel

- **Definition**: use "decibels" as a **logarithmic unit** of measure for a **ratio** between **two powers**

$$\underbrace{10\underbrace{\log_{10}\left(\frac{P_1}{P_2}\right)}_{\text{bel}}}_{\text{decibel}}$$

Next

Know These!!

Decibel Power Rules

Powers of 10 are easy to convert to dB!!

P_1/P_2 (non-dB)	P_1/P_2 (dB)	
$1000 = 10^{3}$	30 dB	**30 dB is "changes power by 1000x"**
$100 = 10^{2}$	20 dB	**20 dB is "changes power by 100x"**
$10 = 10^{1}$	10 dB	**10 dB is "changes power by 10x"**
$1 = 10^{0}$	0 dB	**0 dB is "unity power gain"**
$0.1 = 10^{-1}$	–10 dB	**-10 dB is "changes power by 0.1x"**
$0.01 = 10^{-2}$	–20 dB	**-20 dB is "changes power by 0.01x"**
$0.001 = 10^{-3}$	–30 dB	**-30 dB is "changes power by 0.001x"**

Another "Rule" to Know!!

$P_1/P_2 = 2$ ➔ ~ 3 dB

$P_1/P_2 = 1/2$ ➔ ~ -3 dB

Next

"Extending" the Decibel

- Even though dB is defined for power we can extend it for use with voltages and currents:
  - We "imagine" voltages to be compared are across the same resistance

$$10\log_{10}\left(\frac{P_1}{P_2}\right) = 10\log_{10}\left(\frac{V_1^2/R}{V_2^2/R}\right) = 10\log_{10}\left(\frac{V_1^2}{V_2^2}\right) = 20\log_{10}\left(\frac{V_1}{V_2}\right)$$

$$10\log_{10}\left(\frac{P_1}{P_2}\right) = 10\log_{10}\left(\frac{I_1^2 R}{I_2^2 R}\right) = 10\log_{10}\left(\frac{I_1^2}{I_2^2}\right) = 20\log_{10}\left(\frac{I_1}{I_2}\right)$$

Use "20" for V & I, but use "10" for P

To apply dB to a "power quantity": use 10 $\log_{10}$( )

To apply dB to a "non-power quantity": use 20 log10( )

Next

Know These!!

Decibel Non-Power Rules

V_1/V_2 (non-dB)	V_1/V_2 (dB)
$1000 = 10^{3}$	60 dB
$100 = 10^{2}$	40 dB
$10 = 10^{1}$	20 dB
$1 = 10^{0}$	0 dB
$0.1 = 10^{-1}$	–20 dB
$0.01 = 10^{-2}$	–40 dB
$0.001 = 10^{-3}$	–60 dB

Note: A 10x voltage change is a 10^2 = 100x power change (20 dB)

Another "Rule" to Know!!

$V_1/V_2 = 2$ ➔ ~ 6 dB

$V_1/V_2 = 1/2$ ➔ ~ -6 dB

$V_1/V_2 = \sqrt{2} \Rightarrow \sim 3\text{ dB}$

$V_1/V_2 = 1/\sqrt{2} \Rightarrow \sim -3\text{ dB}$

Next

Sounds	Sound Pressure Level (μBar)	**Sound Pressure Level** (dB)
Jet Plane (@ 30 m)	2000	140
Threshold of Pain		130
	200	120
Chainsaw		110
Rock Concert/Club	20	100
		90
Busy Street	2	80
		70
Normal Speech	0.2	60
		50
	0.02	40
Quiet Room		30
Recording Studio	0.002	20
		10
Threshold of Hearing	0.0002	0

"Reference Level" = 0 dB

Not Power

$\rightarrow 20\log_{10}(SPL/y)$

What should "y" be?

A Reference Level!!

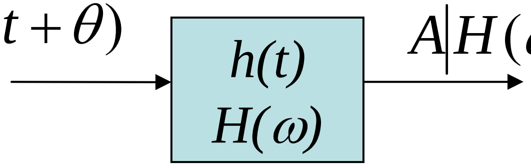
$$20\log_{10}\left(\frac{20}{0.0002}\right)$$

Next

<u>When applying dB to frequency response magnitude use</u>: **20 log10(|*H*(*ω*)|)**

Why? Because $|H(\omega)|$ relates Voltages (or current)!!!

$$A\cos(\omega_0 t + \theta) \quad \longrightarrow \quad A|H(\omega_0)|\cos(\omega_0 t + \theta + \angle H(\omega_0))$$



Input <u>voltage</u> amplitude $= A$

Output <u>voltage</u> amplitude $= A|H(\omega_0)|$

$$|H(\omega_0)| = \frac{\text{Output Voltage Amplitude}}{\text{Input Voltage Amplitude}}$$

Frequency Response Magnitude is <u>Voltage Ratio</u> → **20 log10(|*H*(*ω*)|)**

Next

Methods for Making Bode Plots

It is easy to use computers to make Bode plots… we already saw how to do that!

But good engineers need *insight* to:

- understand the results of an analysis

- make decisions for design

We'll focus on ***insight*** into the magnitude $|H(\omega)|$…

(insight into the phase of $H(\omega)$ can be also be gained through similar steps)

We'll explore several examples of creating Bode Plots…

…we'll fuse it all together into a general method in "Discussion Section Notes"

Later in these notes we'll show how the insight gained from this can be used in a Design Example

Next

Example 1: Bode Plot of a Single Pole (RC Circuit):

Let $a = 1/RC$… then the transfer function of the simple RC circuit is:

$$H(s) = \frac{a}{s+a} \quad \Rightarrow \quad H(\omega) = \frac{a}{j\omega + a}$$

For $a > 0$ the pole is in the “left-half plane” and we can switch to Frequency Response

Single Pole (a is real)

We can write this as:

$$H(\omega) = \frac{1}{1 + j\omega / a}$$

Manipulate like this for “convenience”… we’ll find the (1+j-term) makes things easy when finding approximations.

Now look at the magnitude:

$$|H(\omega)| = \frac{1}{|1 + j\omega / a|}$$

Next

Now focus on <u>approximating</u> |*H*(ω)| for <u>ω > 0</u>

Recall… |*H*(ω)| has even symmetry

$$|H(\omega)| = \frac{1}{|1 + j\omega / a|}$$

Approximate behavior when *ω* << *a*: ➔ *ω/a* << 1

What qualifies? $\omega \leq 0.1a$

$$|H(\omega)| = \frac{1}{|1 + j(small)|} \approx \frac{1}{|1 + j0|} = 1$$

Approximate behavior when *ω* >> *a*: ➔ 1 << *ω/a*

What qualifies? $\omega \geq 10a$

$$|H(\omega)| = \frac{1}{|(small) + j\omega / a|} \approx \frac{1}{|j\omega / a|} = \frac{a}{|j||\omega|} = \frac{a}{|\omega|}$$

So we now know: $|H(\omega)| = \frac{1}{|1 + j\omega / a|}$ ➔ $|H(\omega)| \approx \begin{cases} 1, & \omega \leq 0.1a \\ \frac{a}{|\omega|}, & \omega \geq 10a \end{cases}$

Next

Now convert this approximation into dB form:

$$20\log_{10}\{|H(\omega)|\} = 20\log_{10}\left\{\frac{1}{|1+j\omega/a|}\right\} = \underbrace{20\log_{10}\{1\}}_{=0} - 20\log_{10}\left\{\left|1+\frac{j\omega}{a}\right|\right\}$$

So our Bode plot (dB vs. $\log_{10}\omega$) is: $-20\log_{10}\left\{\left|1+\frac{j\omega}{a}\right|\right\}$ vs. $\log_{10}\omega$

From our approximations above this is yields:

For $\omega << a$: 1 vs. $\log_{10}\omega$ → **Constant ⇒ flat line @ 0 dB**

For $\omega >> a$: $\underbrace{A - 20[\log_{10}\omega] \text{ vs. } [\log_{10}\omega]}_{(A-20x) \text{ vs. } x}$ → Line of slope -20

↓ Drops 20dB for every unit change in $\log_{10}\omega$

↓ Drops 20dB for every 10x change in ω

↓ **Line with slope of "-20dB per decade"**

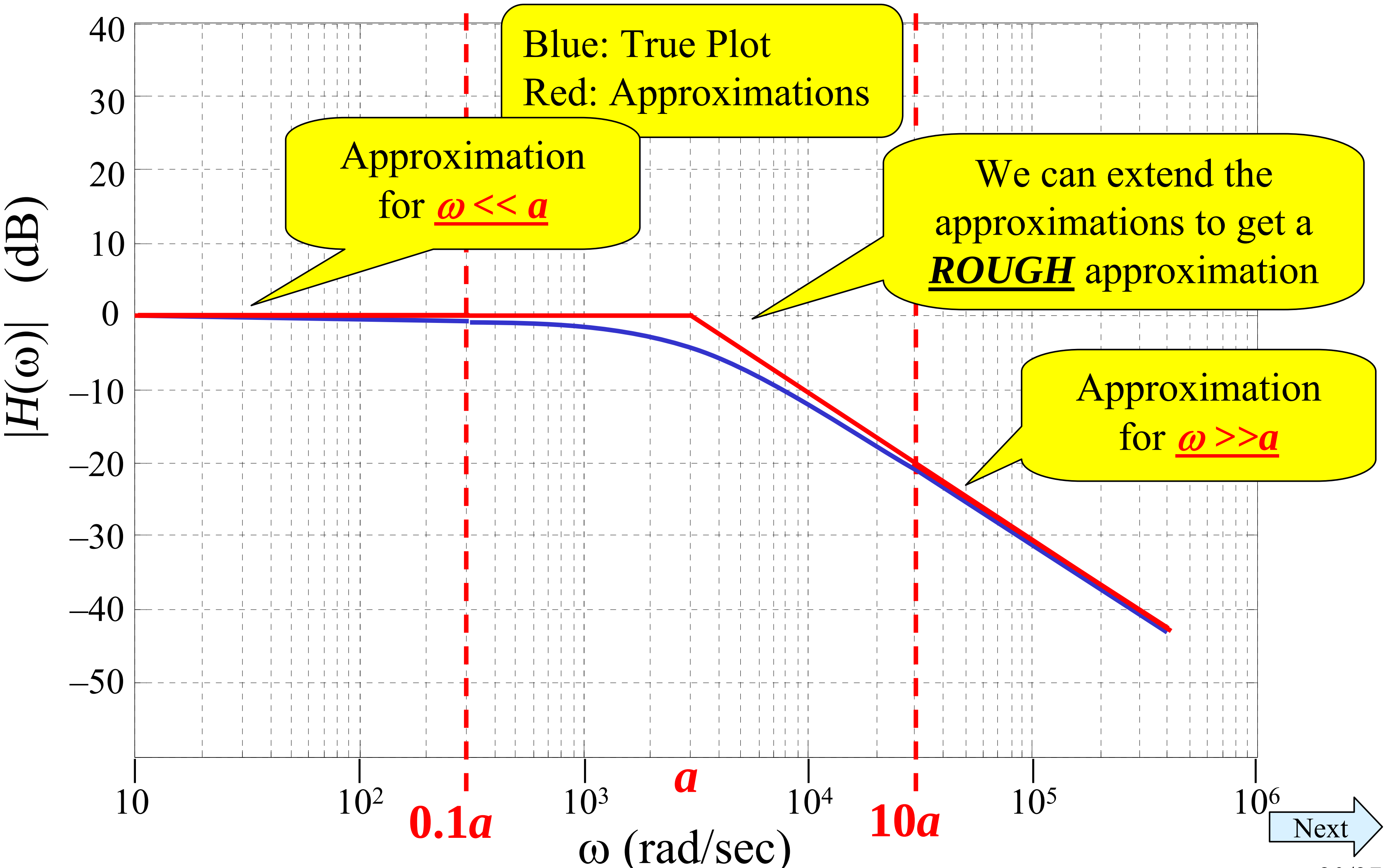
Note:
-20dB/decade
= -6dB/octave

Next

Approximations vs. True Plot

$a = 1/RC = 3000$ rad/sec

$$H(s) = \frac{a}{s+a}$$



Next

Quick steps for making a Bode Plot for a "Single Pole Term"

Now that we've done all this work once we can now just quickly make the Bode plot for this type of $H(s)$:

$$H(s) = \frac{a}{s+a}$$

1. Find $\omega = a$ on the ω-axis
2. Draw a horizontal line at 0 dB from left-edge up to $\omega = a$
3. Go "over 1 decade" and "down 20 dB" … draw a point
4. Draw a sloped line from the end of the first line through this point

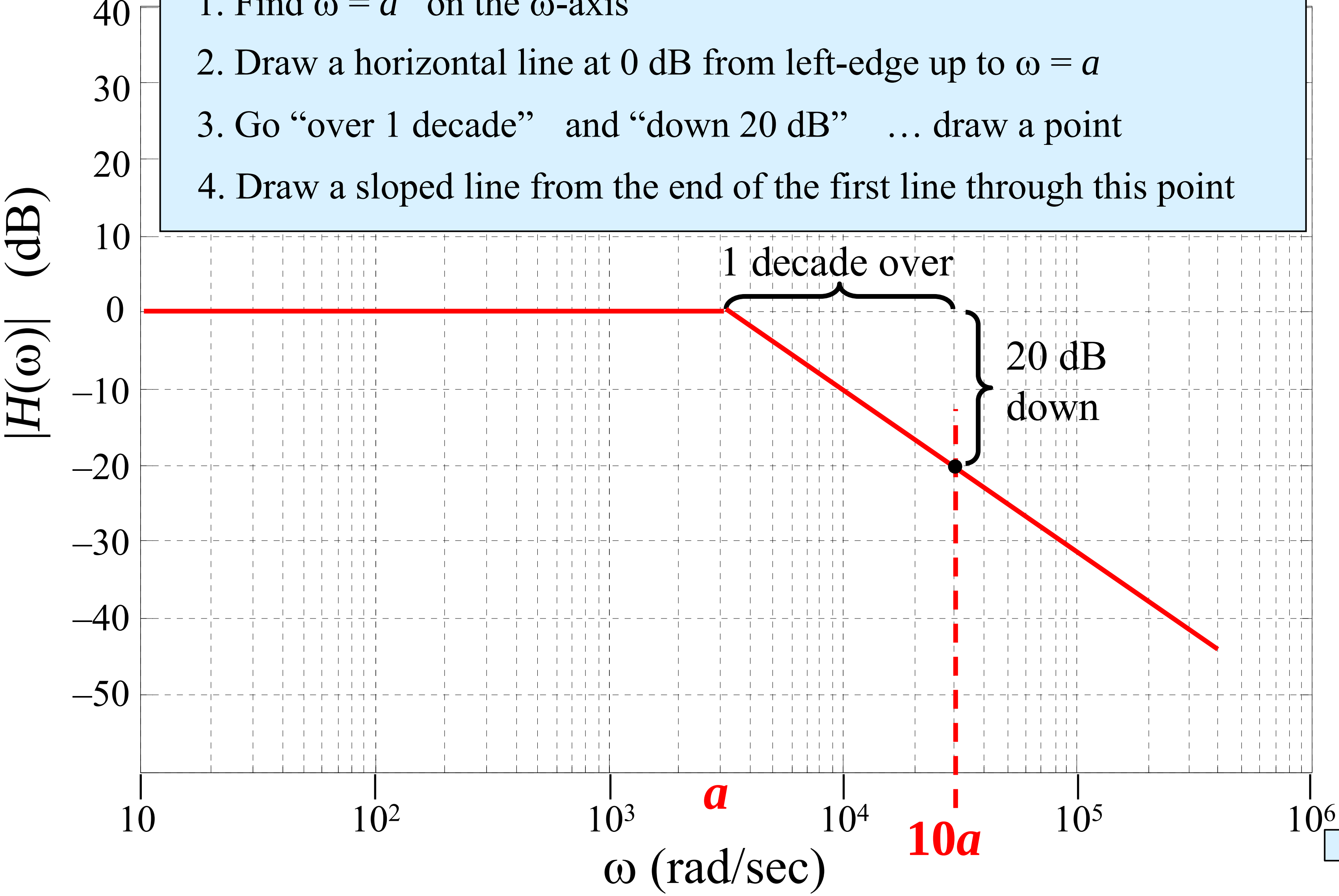
The steps are illustrated on the next slide….

Approximate Bode Plot for Single Real Pole

$a = 1/RC = 3000$ rad/sec

$$H(s) = \frac{a}{s+a}$$

1. Find $\omega = a$ on the ω-axis
2. Draw a horizontal line at 0 dB from left-edge up to $\omega = a$
3. Go "over 1 decade" and "down 20 dB" … draw a point
4. Draw a sloped line from the end of the first line through this point



Next

Example 2: Bode Plot of a Single Zero:

Now… if instead we had:

$$H(s) = \frac{s+a}{a} \Rightarrow H(\omega) = \frac{j\omega + a}{a} \Rightarrow H(\omega) = 1 + j\omega / a$$

Using the same kind of steps…

Our Bode plot (dB vs. $\log_{10}\omega$) is: $+20\log_{10}\left\{\left|1 + \frac{j\omega}{a}\right|\right\}$ vs. $\log_{10}\omega$

Note: we have "+" here now

Similarly we arrive at our approximations:

For $\omega << a$: 1 vs. $\log_{10}\omega$ ⇒ **Constant ⇒ flat line @ 0 dB**

For $\omega >> a$: A+20[$\log_{10}\omega$] vs. [$\log_{10}\omega$] ⇒ Line of slope +20 ⇒ **Line with slope of "+20dB per decade"**

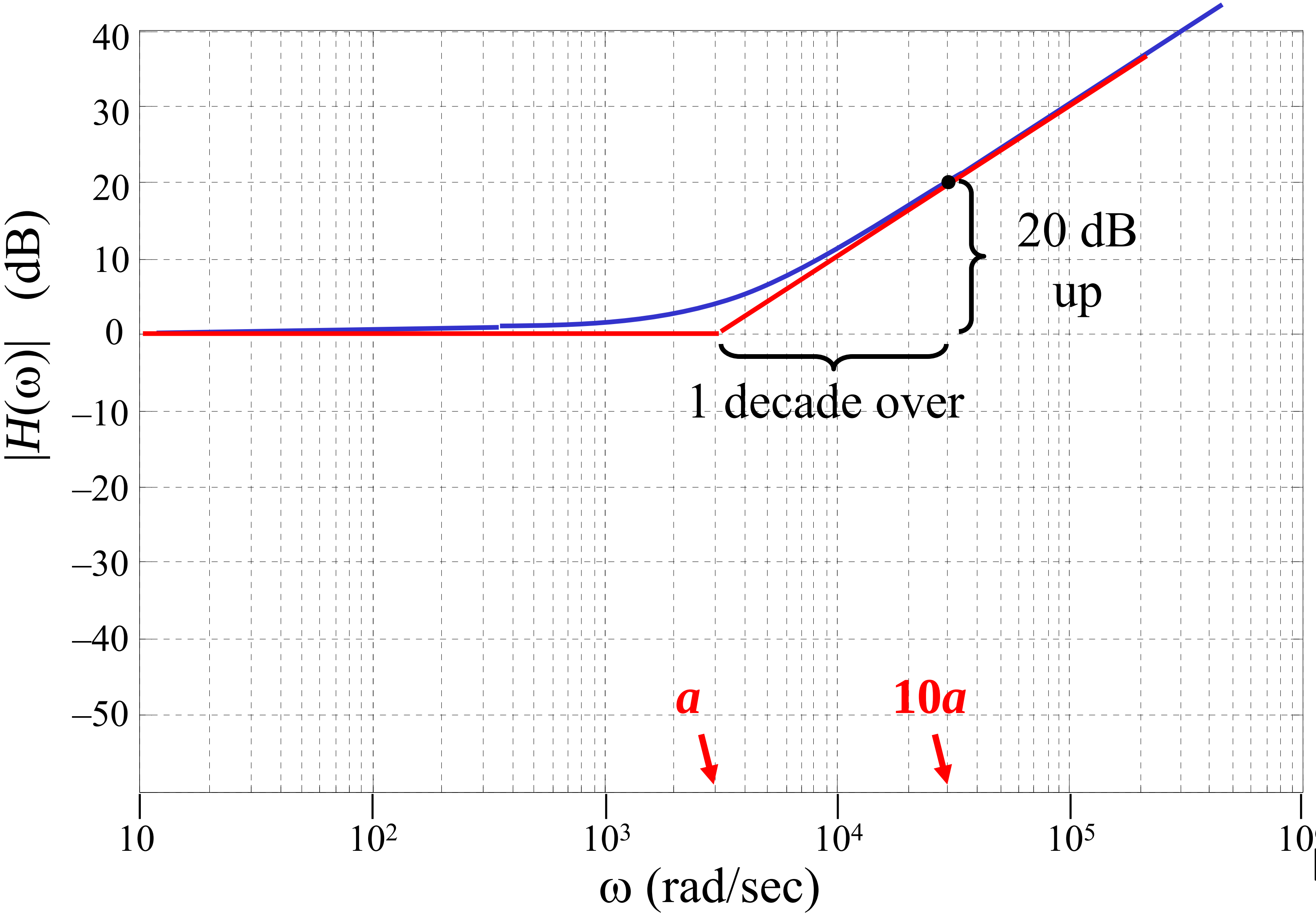
Note: we have "+" here now

Next

Approximate Bode Plot for Single Real Zero

$a = 3000$ rad/sec

$$H(s) = \frac{s+a}{a}$$



Next

Example 3: Bode Plot of a Single Zero & Single Pole:

$$H(s) = \frac{a(s+b)}{b(s+a)} \;\rightarrow\; H(\omega) = \frac{a(b+j\omega)}{b(a+j\omega)} \;\rightarrow\; H(\omega) = \frac{(1+j\omega/b)}{(1+j\omega/a)}$$

Following the same steps, our Bode plot is:

$$\left[+20\log_{10}\left\{\left|1+\frac{j\omega}{b}\right|\right\} + \left(-20\log_{10}\left\{\left|1+\frac{j\omega}{a}\right|\right\}\right)\right] \quad \text{vs.} \quad \log_{10}\omega$$

Just… add the two types of Bode plots we've already seen!

Recall: log(AB) = log(A) + log(B)

log(A/B) = log(A) - log(B)

Note: (line #1) + (line #2) = line with slope of (sum of slopes)

"Slopes add"

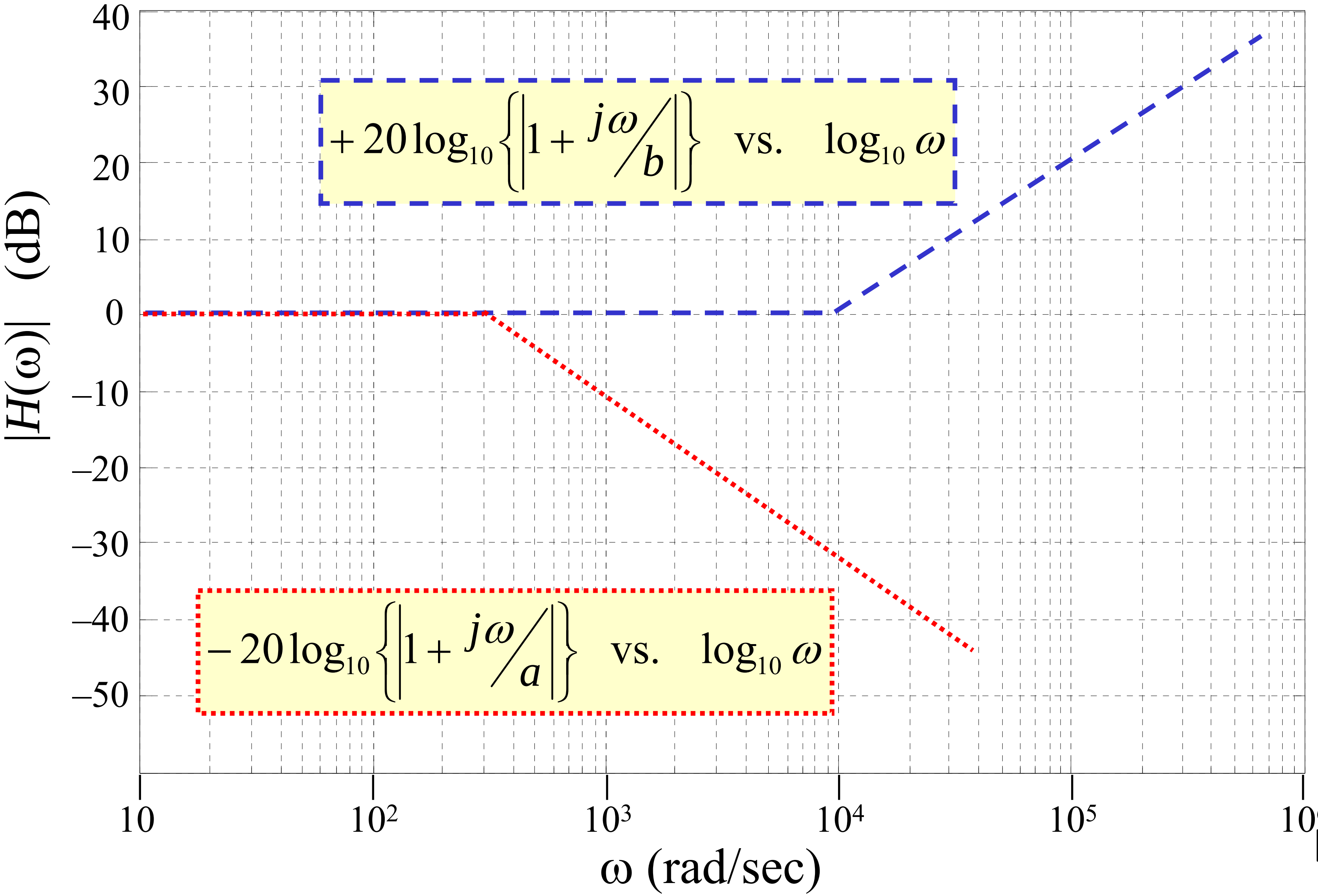
$$\left(m_1x+b_1\right)+\left(m_2x+b_2\right) = \left(m_1x+m_2x\right)+\left(b_1+b_2\right)$$
$$= \left(m_1+m_2\right)x+\left(b_1+b_2\right)$$

Next

Shows the 2 Parts that Add

$a = 300$ rad/sec $\quad b = 10000$ rad/sec

$$H(s) = \frac{a(s+b)}{b(s+a)}$$

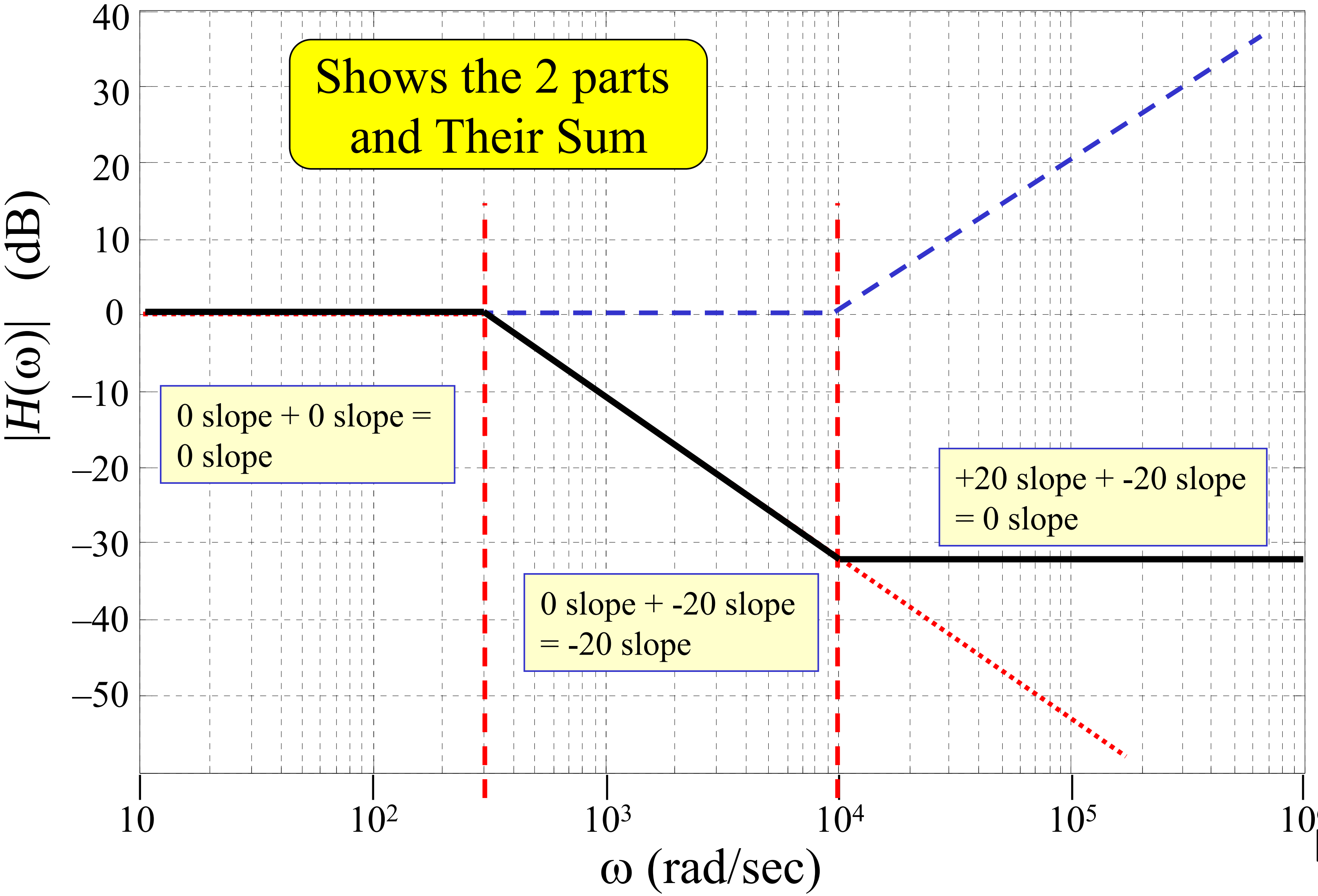


Next

Approx. Bode Plot for One Real Zero & One Real Pole

$a = 300$ rad/sec $\qquad b = 10000$ rad/sec

$$H(s) = \frac{a(s+b)}{b(s+a)}$$

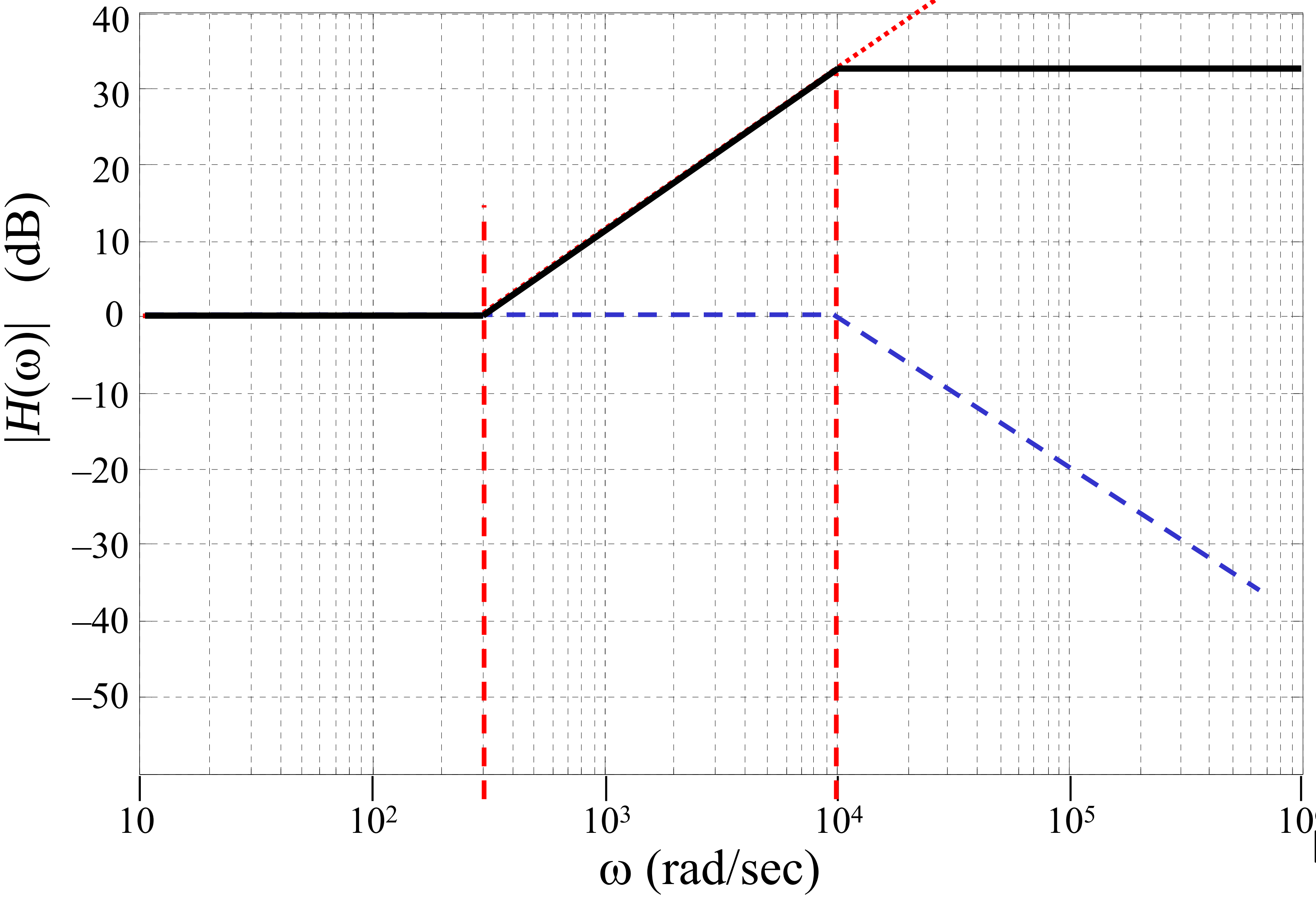


Next

Same Structure but "zero first, then pole"

$a = 10000$ rad/sec $b = 3000$ rad/sec

$$H(s) = \frac{a(s+b)}{b(s+a)}$$



Next

Summary of Bode Plot Examples so Far (only real poles & zeros)

1. We can use simple approximations and graphing tricks to create a rough approximation to the magnitude of the Frequency Response
2. A first-order pole (not a complex-pair) will cause the Bode plot to
   a. "Break" at "the pole frequency"
   b. Decrease at -20 dB/decade above "the pole frequency"
3. A first-order zero (not a complex-pair) will cause the Bode plot to
   a. "Break" at "the zero frequency"
   b. Increase at +20 dB/decade above "the zero frequency"

Now, given any $H(s)$ with only real zeros and poles we can easily extend these ideas:

$$H(s) = K\frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)} = \underbrace{\frac{Kz_1z_2}{p_1p_2}}\ \underbrace{\frac{(1+s/z_1)(1+s/z_2)}{(1+s/p_1)(1+s/p_2)}}$$

Just moves the <u>whole</u> plot up/down by $20\log_{10}(Kz_1z_2/p_1p_2)$

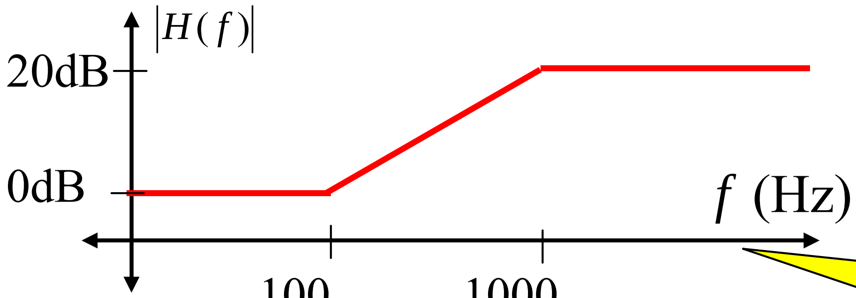
Each term is 0dB at "small enough" ω

Next

Design Example using Bode Plot Insight

Suppose you want to build a "treble booster" for an electric guitar.

You decide that something like this might work:



$100\text{Hz} \Rightarrow 628 \text{ rad/s} = \omega$

$1000\text{Hz} \Rightarrow 6283 \text{ rad/s} = \omega$

Notice that we are doing our rough "design thinking" in terms of Bode Plot approximations!!!

The A string on a guitar has a fundamental frequency of 110 Hz
The A note on 17th fret of the high-E string has a fundamental frequency of 880 Hz

From our Bode Plot Insight… we know we can get this from a single real pole, single real zero system… with the "zero first, then the pole":

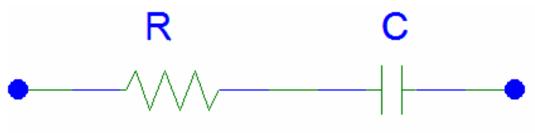
$$H(s) = \frac{(1 + s/\omega_1)}{(1 + s/\omega_2)} \quad \Rightarrow \quad H(\omega) = \frac{(1 + j\omega/\omega_1)}{(1 + j\omega/\omega_2)}$$

with: $\omega_1 = 628$ rad/s
$\omega_2 = 6280$ ra

Next

Now, how do we get a circuit to <u>do</u> this? Let's explore!

A series combination



…has impedance $Z(s) = R + 1/Cs$

Note: we could get $R + sL$ with an inductor but inductors are generally avoided when possible

So what do we get if we could some how form a ratio of such impedances?

$$\frac{Z_1(s)}{Z_2(s)} = \frac{R_1 + 1/C_1 s}{R_2 + 1/C_2 s} = \frac{C_2}{C_1}\frac{(sR_1C_1 + 1)}{(sR_2C_2 + 1)}$$

Aha!!! What we want!

$\Rightarrow$ Let: $\omega_1 = 1/R_1C_1$

$\omega_2 = 1/R_2C_2$

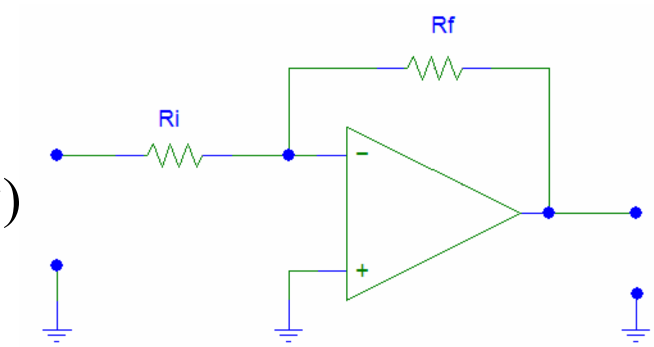
$$\frac{Z_1(\omega)}{Z_2(\omega)} = \frac{C_2}{C_1}\frac{(1 + j\omega/\omega_1)}{(1 + j\omega/\omega_2)}$$

Next

Okay…how do we build a circuit that has a transfer function that is a ratio of impedances?!

Recall the op-amp inverting amplifier!

$v_{in}(t)$



$v_o(t)$

$$Gain = -\frac{R_f}{R_i}$$

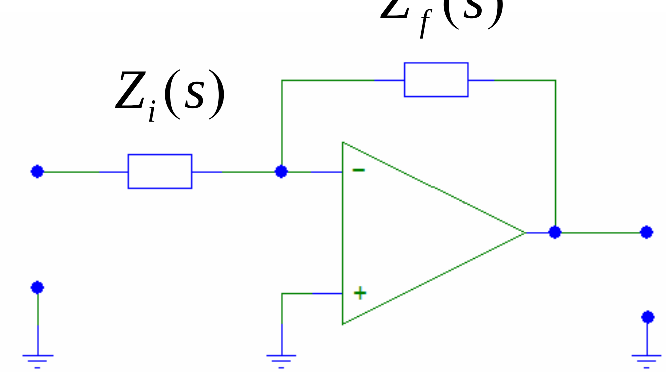
Ratio of **resistances**

Extending the analysis to include impedances we can show that:

$Z_f(s)$

$Z_i(s)$

$v_{in}(t)$

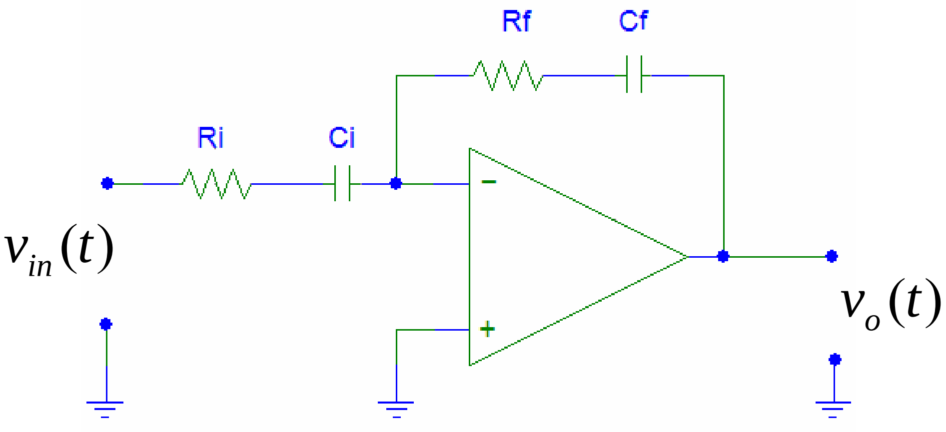


$v_o(t)$

$$\Rightarrow \quad H(s) = -\frac{Z_f(s)}{Z_i(s)}$$

Won't affect our magnitude: $|H(\omega)|$

Next



Now, you can choose the R's & C's to give the desired frequency points

p. 98, *The Art of Electronics*, Horowitz & Hill, Cambridge Press, 1980

But wait!! You then remember that op amps must always have negative feedback at DC so putting C_f here is not a good idea…

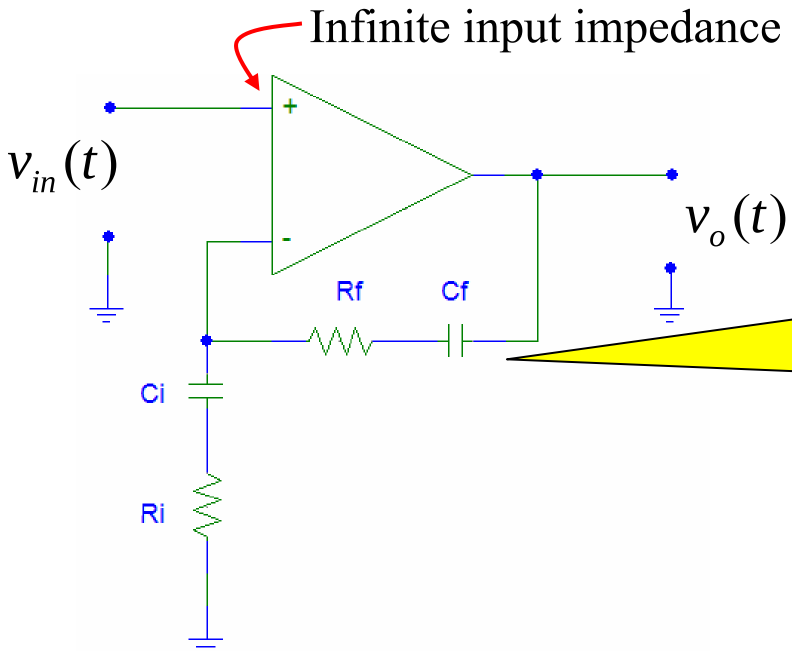
So we have to continue…

We also might not like this circuit because it might not give us a very large input impedance… and that might excessively "load" the circuit that you plug into this (e.g., the guitar)

Back to the drawing board!!!

Next

Okay, then you remember there is also Non-Inverting Op-Amp circuit…



$$Gain = 1 + \frac{R_f}{R_i}$$

We still have the "no DC feedback" issue… but let's charge on and see what might happen!!

Applying this gain formula with the series impedance we get:

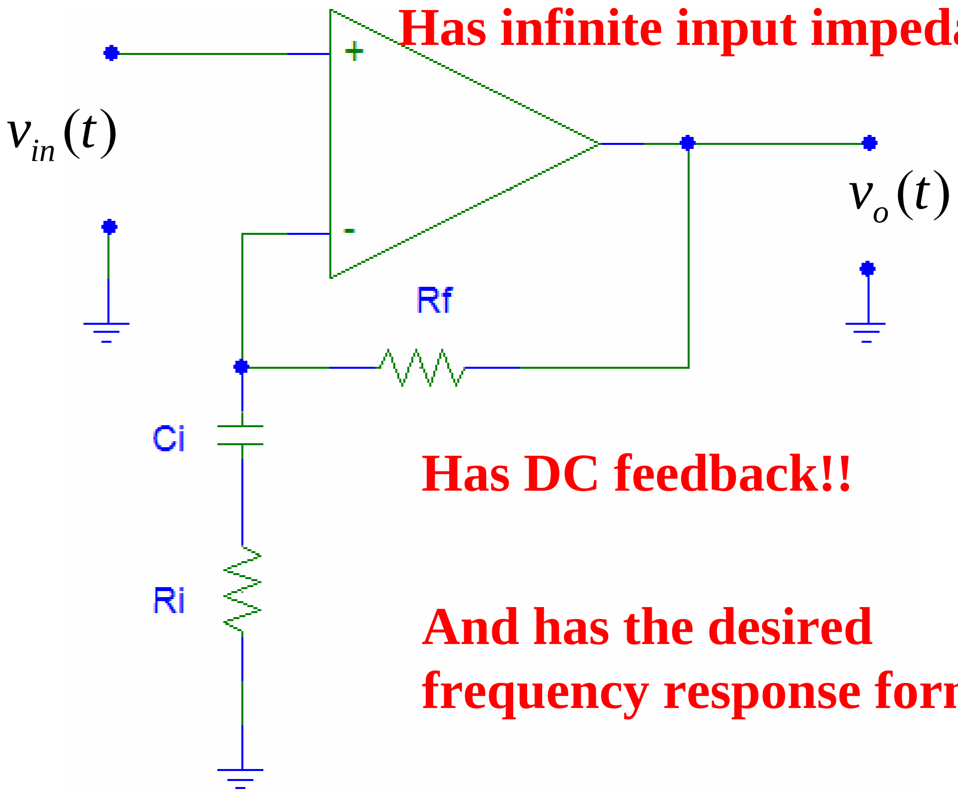
$$H(s) = 1 + \frac{R_f + 1/C_f s}{R_i + 1/C_i s} = \frac{(R_i + R_f) + (1/C_f s + 1/C_i s)}{R_i + 1/C_i s}$$

Oh Cool!! We don't need both of these caps!

Let $C_f = 0$ (replace with "wire")
This fixes our "feedback at DC" issue

Next

So now we have a circuit like this:



Has infinite input impedance!!

Has DC feedback!!

And has the desired frequency response form:

$$H(s) = \frac{(R_f + R_i)C_i s + 1}{R_i C_i s + 1}$$

<u>Set</u>:

$$\omega_1 = \frac{1}{(R_i + R_f)\,C_i} = 628 \text{ rad/s}$$

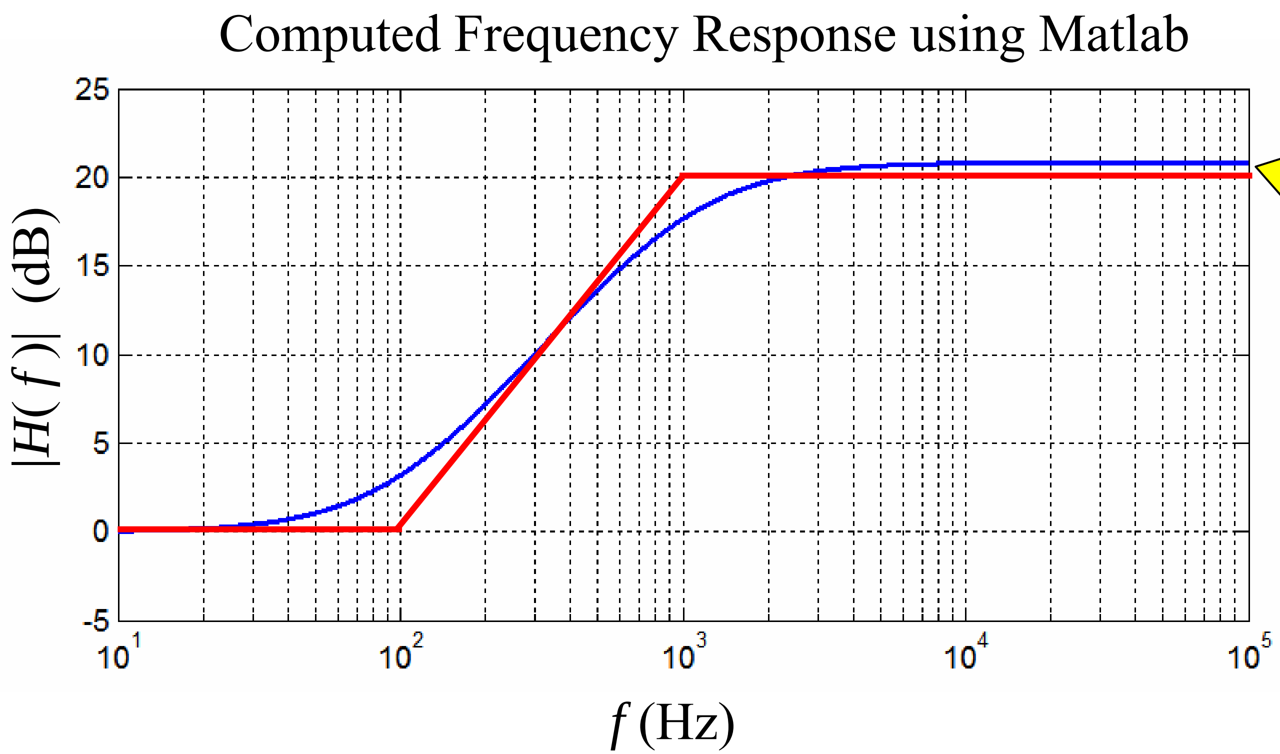
$$\omega_2 = \frac{1}{R_i C_i} = 6283 \text{ rad/s}$$

<u>Choose</u>:

$$R_f = 15k\Omega \qquad R_i = 1.5k\Omega$$

$$C_i = 0.1\mu\text{F}$$

Using standard values



Discrepancy due to use of standard values

Next

Summary of Bode-Plot-Driven Design Example

1. Through insight gained from knowing how to do Bode plots by hand… we recognized the kind of transfer function we needed

2. Through insight gained in circuits class about impedances we recognized a key building block needed: Series R-C

3. Through insight gained in electronics class about op-amps we found a possible solution… the inverting op-amp approach

4. We then scrutinized our design for any overlooked issues
   a. We discovered two problems that we needed to fix

5. We used further insight into op-amps to realize that we could fix the input impedance issue using a non-inverting form of the op-amp circuit

6. We didn't give up at first sign that we might still have the "no DC feedback" problem…
   a. Through mathematical analysis we showed that we could remove the feedback capacitor without changing the circuit's function!!!!!!

Stop