BINGHAMTON UNIVERSITY
State University of New York

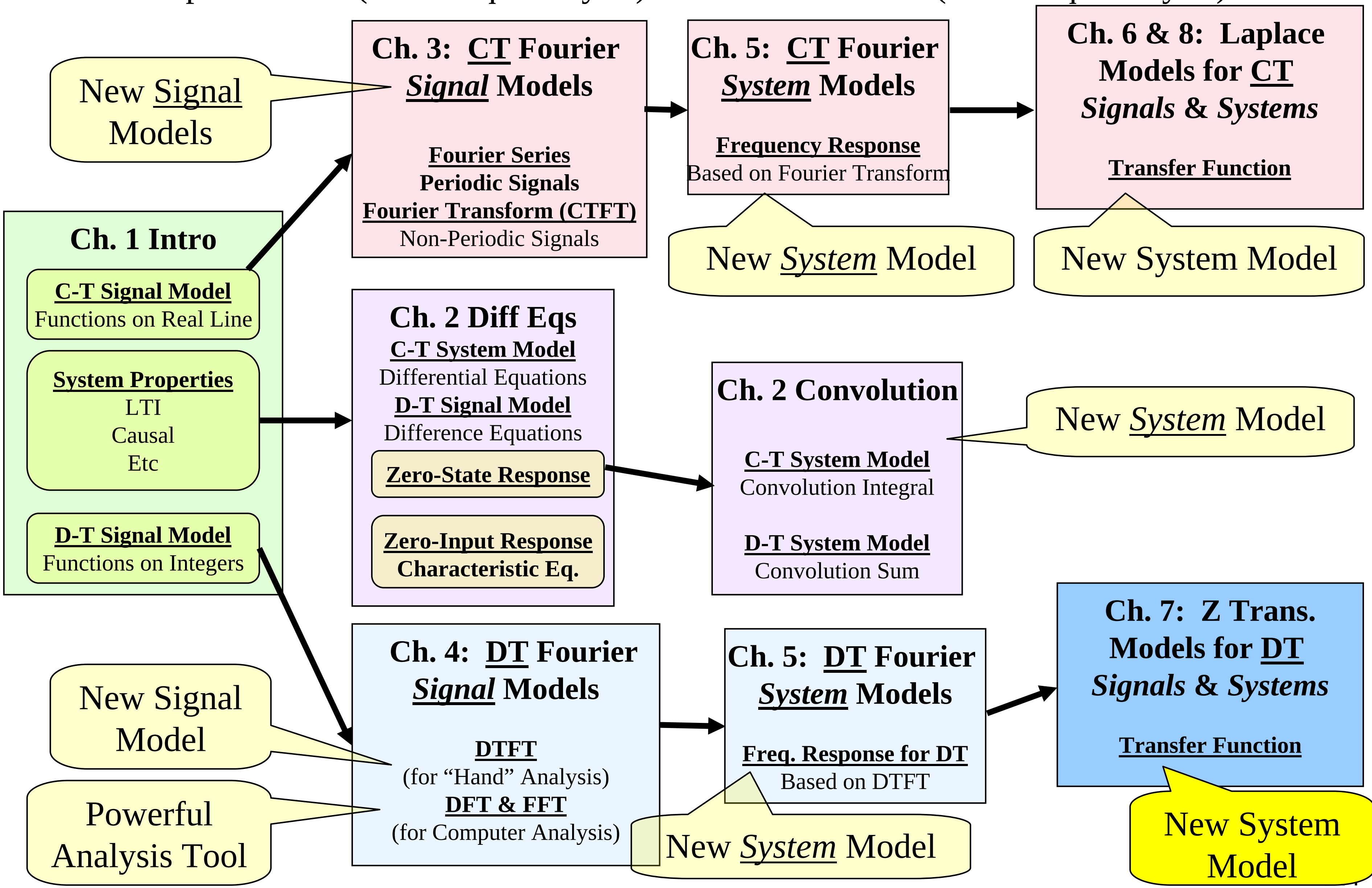
EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #35

- D-T Systems: Z-Transform … Stability of Systems, Frequency Response
- Reading Assignment: Section 7.5 of Kamen and Heck

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



Stability of DT Systems

For systems with rational $H(z)$:

$$\text{It is "BIBO" stable if } \sum_{n=0}^{\infty} |h[n]| < \infty$$

$$\text{Recall: } H(z) = \frac{B(z)}{A(z)}$$

$$\text{Where } A(z) = z^N + a_1 z^{N-1} + \ldots + a_{N-1} z + a_N$$

Any common roots in $B(z)$ and $A(z)$ are assumed to have been cancelled.

$$\text{Let } A(z) \text{ have roots of } \underbrace{p_1, p_2, \ldots, p_N}_{\underline{\text{poles}} \text{ of } H(z)}$$

$$\text{Then } H(z) = \frac{B(z)}{(z - p_1)(z - p_2)\ldots(z - p_N)}$$

and

$$h[n] = h_1[n] + h_2[n] + \ldots + h_N[n]$$

Note: each $h_1[n]$ will have $(p_i)^n u[n]$ decays if $|p_i| < 1$

Result:

$$\boxed{\sum_{n=0}^{\infty} \left| h[n] \right| < \infty}$$

$\Downarrow$

System is stable

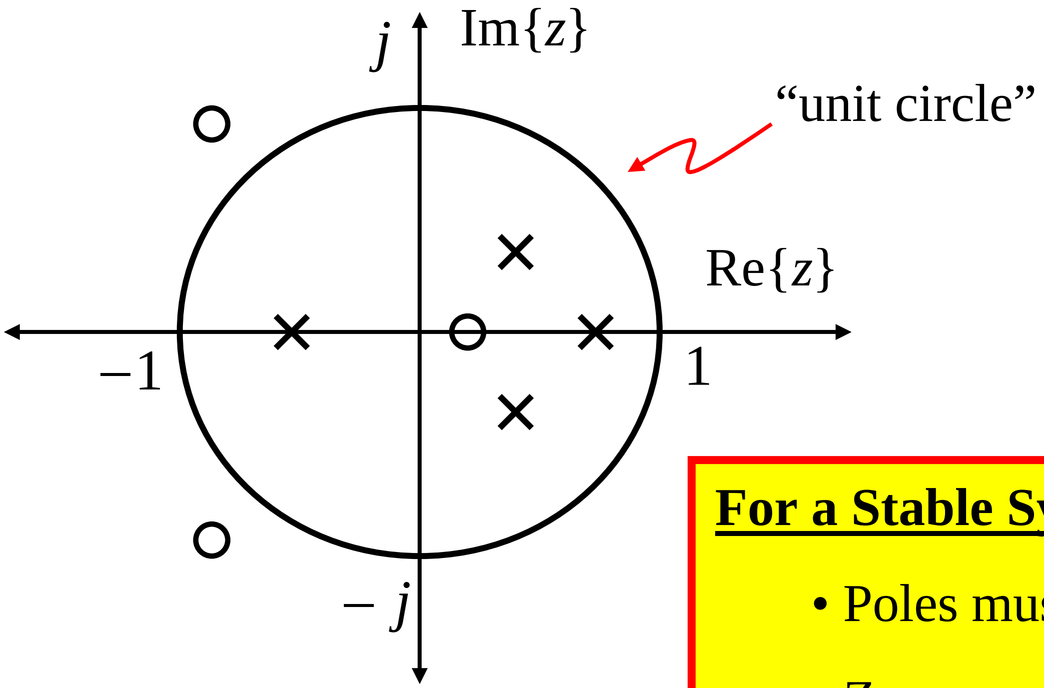
is equivalent to

$$\boxed{all \ \left| p_i \right| < 1}$$

i.e., poles are inside the unit circle

$\Downarrow$

System is stable



For a Stable System

- Poles must be "inside unit circle"
- Zeros can be anywhere

Aside: Complex poles and complex zeros must occur as conjugate pairs

Frequency Response

All the same as for the CT case!   (e.g. how sinusoids go through, how general signals go through)

$$\boxed{H(\Omega) = \sum_{n=0}^{\infty} h[n] e^{-j\Omega n} \quad = H(z)\Big|_{z=e^{j\Omega}}}$$

Using Matlab to Compute Frequency Response:

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_N z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

Some b_i may be 0

Some a_i may be 0

$>>$ num $= [b_0\ b_1 \ldots b_N]$ must put any zero b_i into the vector

$>>$ den $= [a_0\ a_1 \ldots a_N]$ must put any zero a_i into the vector

$>>$ omega = -pi : ? : pi   Pick appropriate spacing

$>>$ $\boxed{\text{H = freqz(num, denom, omega)}}$

$>>$ plot(omega/pi, abs(H))

$>>$ plot(omega/pi, angle(H))

Relationship between the ZT and the DTFT

Recall: $\boxed{H(\Omega) = H(z)\big|_{z=e^{j\Omega}}}$

Let's explore this idea with some pictures for an explicit case…

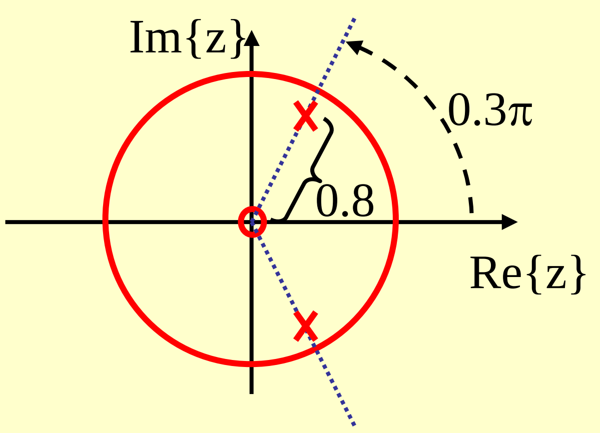
Consider the Z-Transform given by:

This causes $H(z) = 0$ for $z = 0$

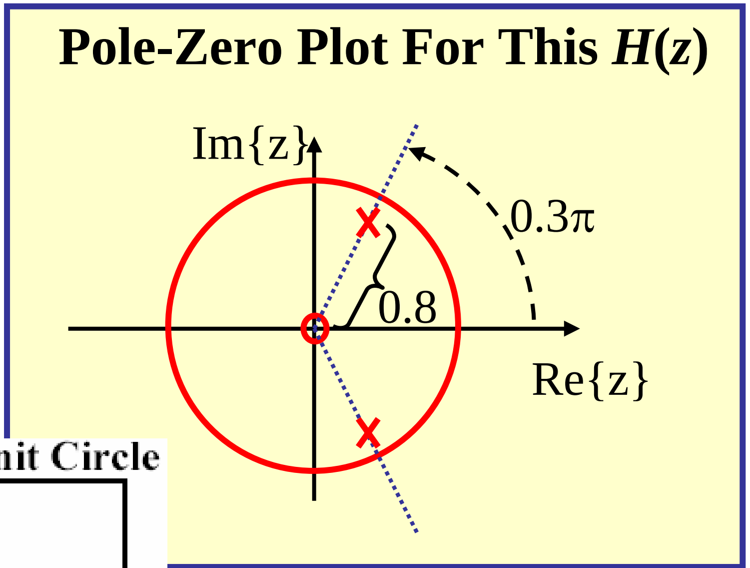
$$H(z) = \frac{1}{\left(1 - 0.8e^{j0.3\pi}z^{-1}\right)\left(1 - 0.8e^{-j0.3\pi}z^{-1}\right)} = \frac{z}{\left(z - 0.8e^{j0.3\pi}\right)\left(z - 0.8e^{-j0.3\pi}\right)}$$

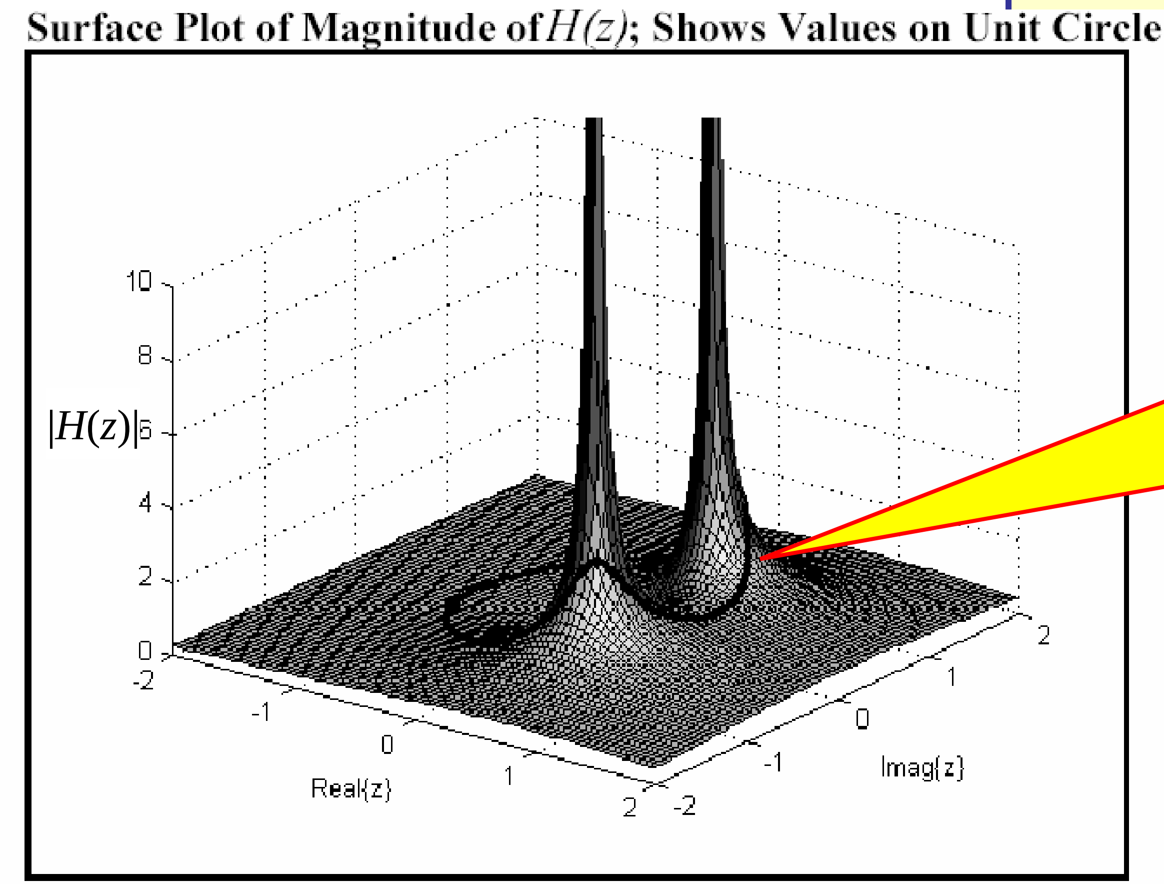
These cause $H(z) = \infty$ for $z = 0.8e^{\pm j0.3\pi}$

Pole-Zero Plot For This *H*(*z*)



So… from this pole-zero plot we can then imagine that the plot of the $|H(z)|$ might look something like this:

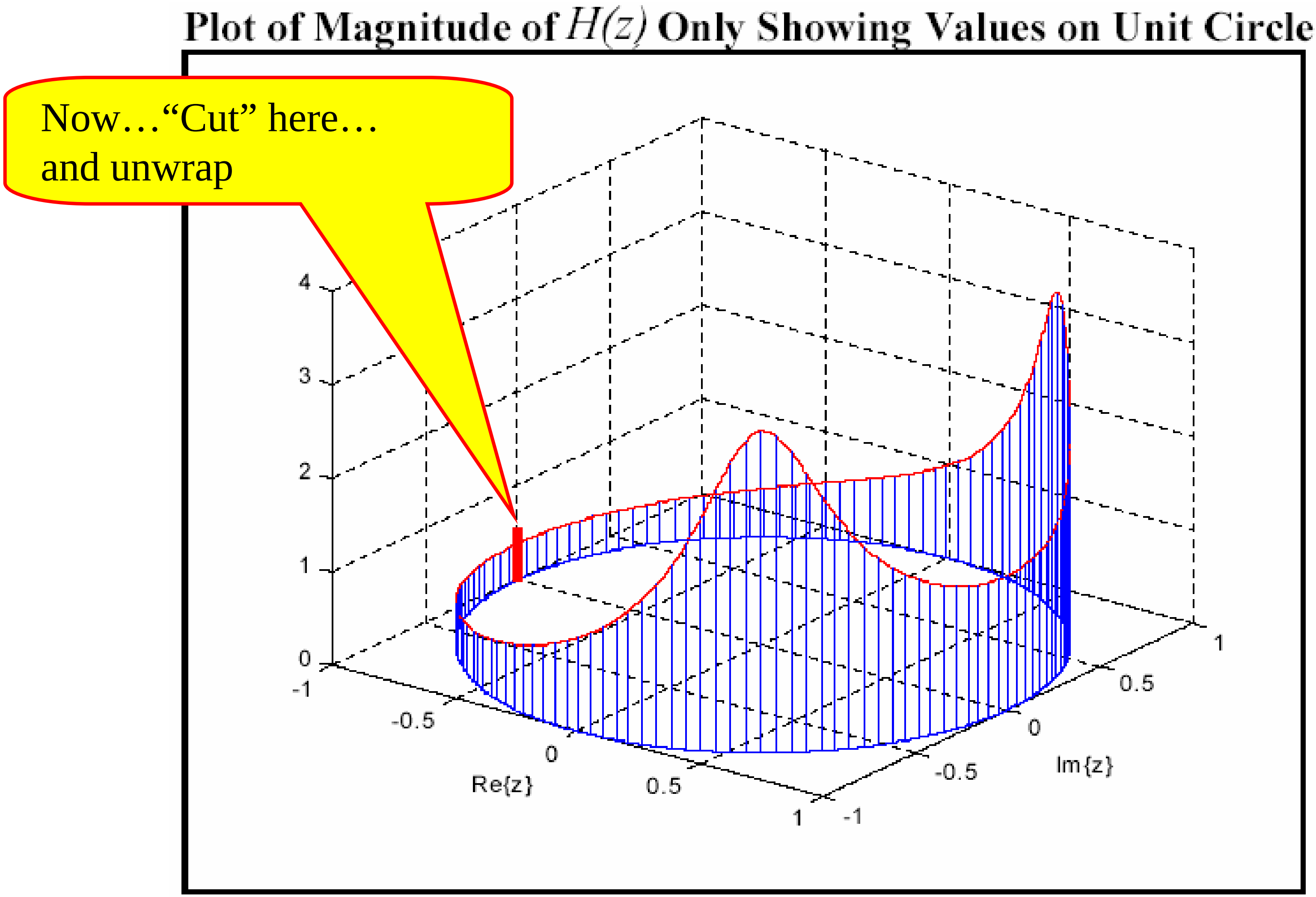




And we know that the Frequency Response is just the Transfer Function evaluated on the Unit Circle.

$$\boxed{H(\Omega) = H(z)\Big|_{z=e^{j\Omega}}}$$

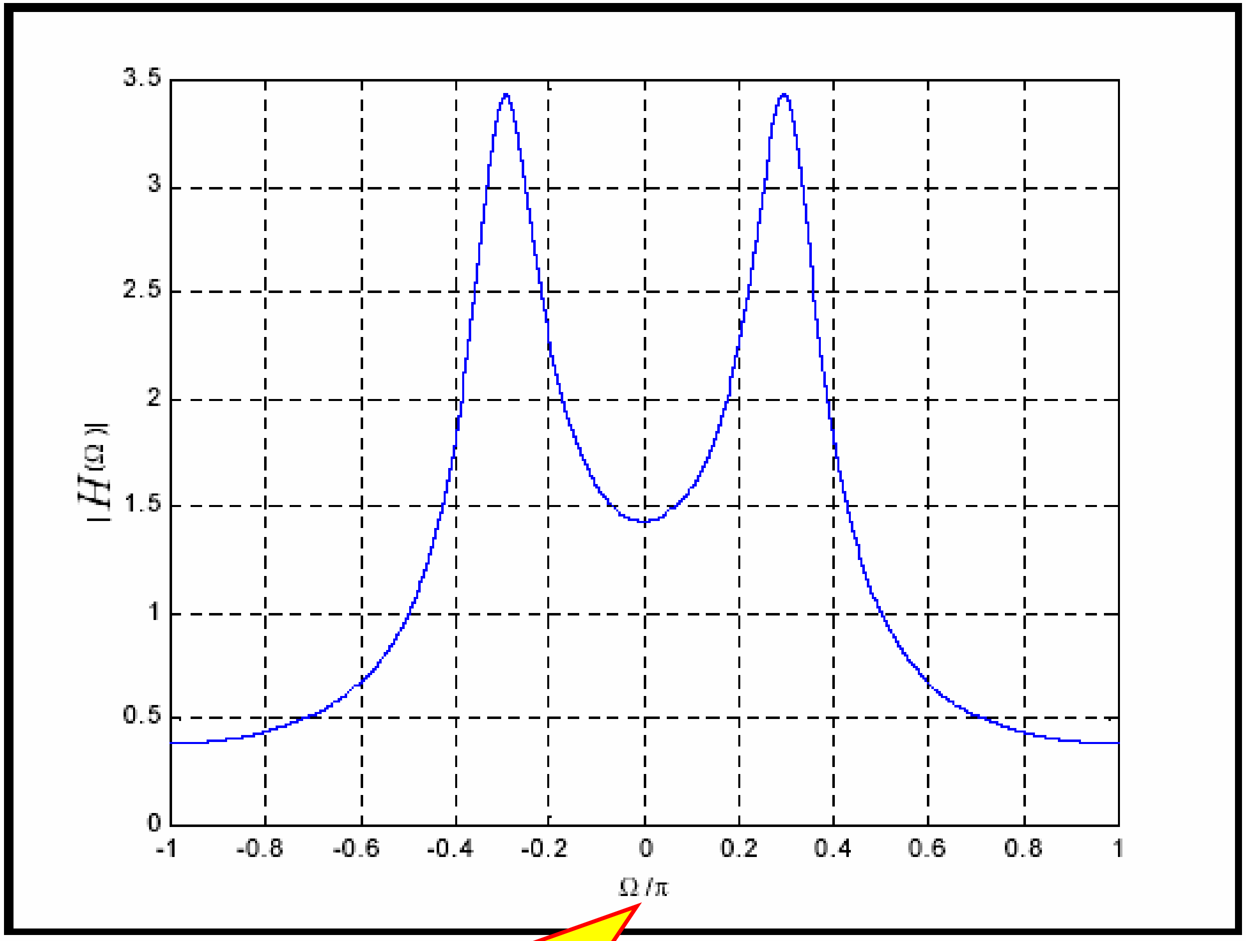
Now… plot just those values on the unit circle:



This shows the Frequency Response $H(\Omega)$ where Ω is the angle around the unit circle… this explains why $H(\Omega)$ is a periodic function of Ω

This shows the previous plot "cut and unwrapped"…
and plotted on the Ω axis:

DTFT = ZT on Unit Circle

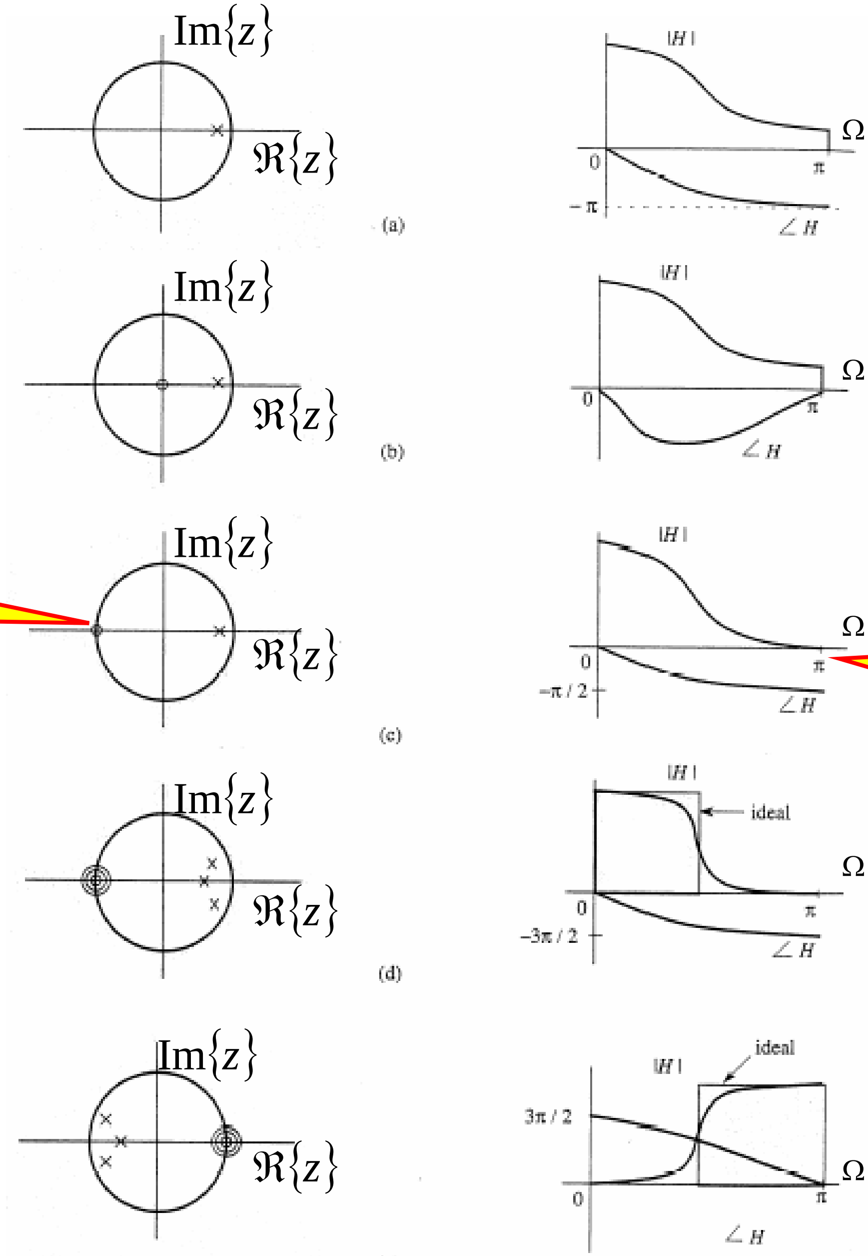


Normalized for convenience

Effect of Poles & Zeros on Frequency Response of DT filters

Note: Including a pole or zero at the origin …

…doesn't change the magnitude but does change the phase



Placing a zero at ±π…

…makes |H(π)| = 0

Placing more zeros/poles…

… gives sharper transitions.

Figure from B.P. Lathi, Signal Processing and Linear Systems

So… from these plots and ideas we can see that we could design simple DT filters by deciding where to put poles and zeros.

This is not a very good design approach…

… but this insight is crucial to understanding transfer functions.

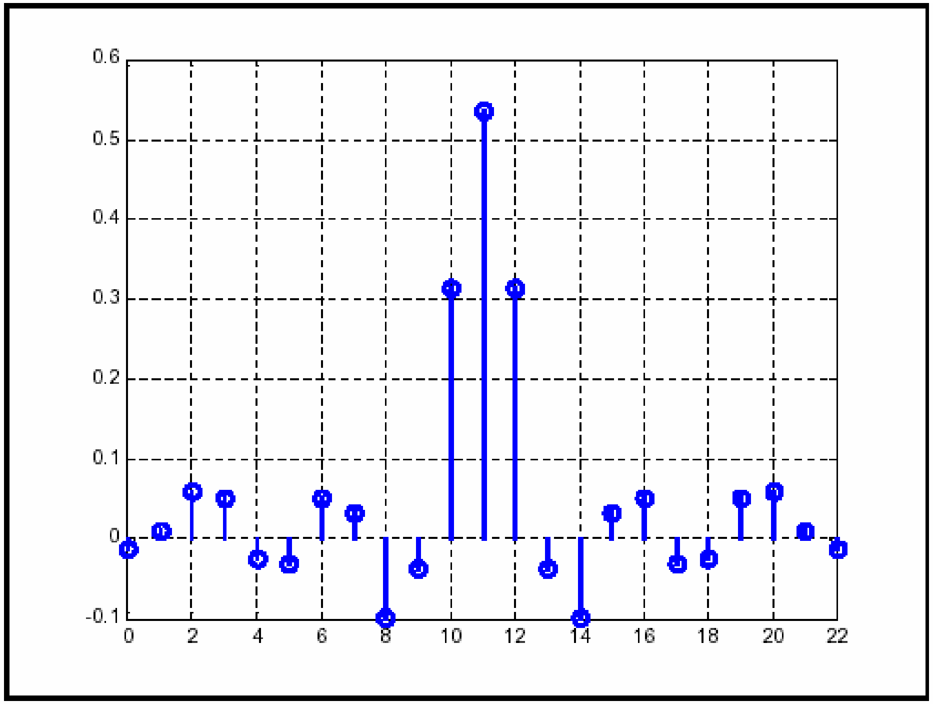
The following charts in this set of notes shows a filter designed not by placing poles and zeros but rather by using one really good computer-based design method for designing DT filters.

A practical DT filter (Designed using MATLAB's remez)

(See Digital Signal Processing course to learn the design process)

Here is the impulse response $h[n]$… it is assumed to be zero where not shown…

Note that it has only finite many non-zero samples



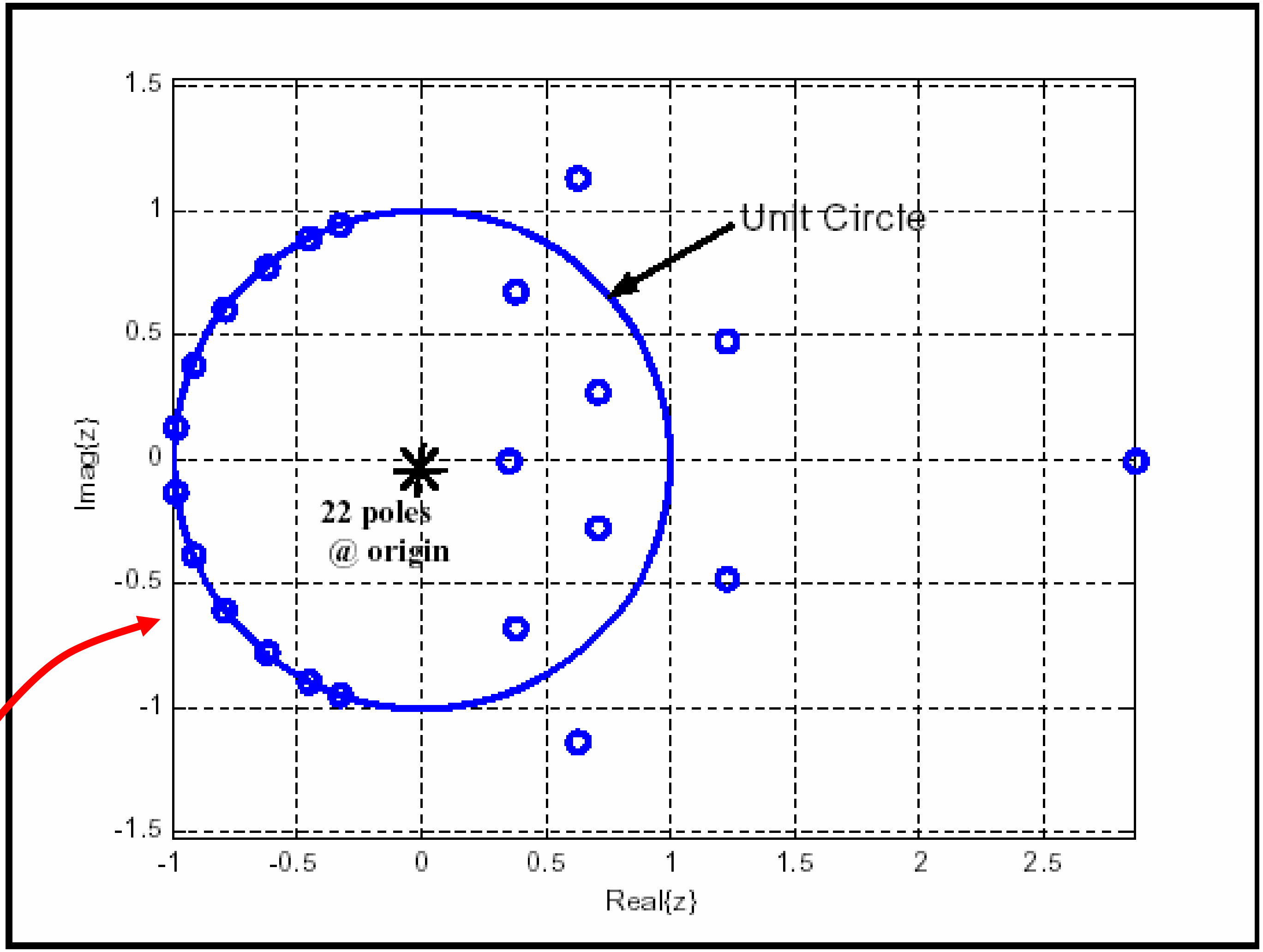
Called a "Finite-impulse Response" (FIR) filter

$$H(z) = Z\{h[n]\} = h[0] + h[1]z^{-1} + h[2]z^{-2} + \cdots + h[21]z^{-21} + h[22]z^{-22}$$

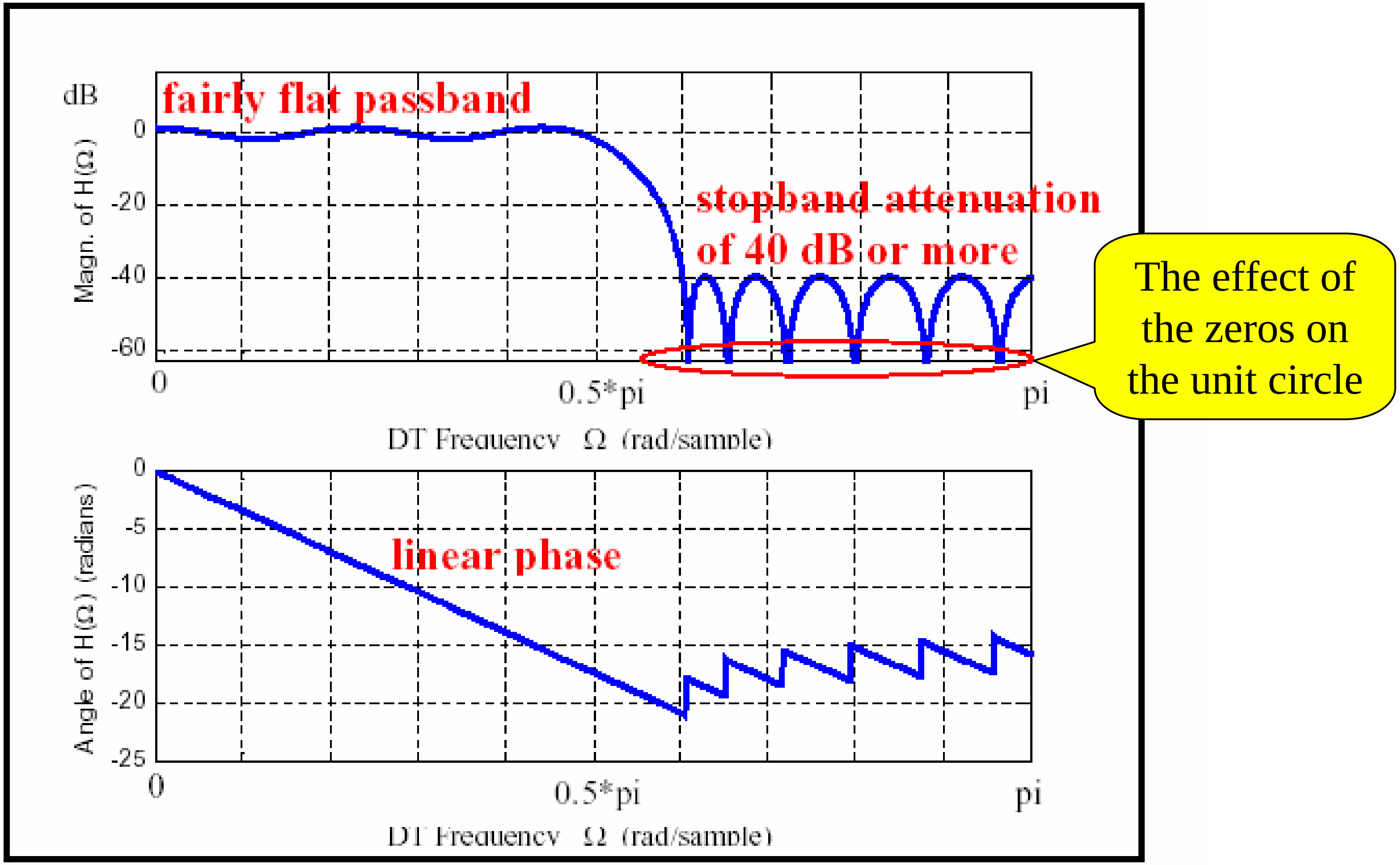
$$= \frac{h[0]z^{22} + h[1]z^{21} + h[2]z^{20} + \cdots + h[21]z^{1} + h[22]}{z^{22}}$$

22 zeros

22 poles at origin



All these zeros, right on the unit circle, pull the frequency response down to create the stop band



Note, this filter has linear phase in the passband… this is the ideal phase response (as we saw back in Ch. 5 for CT filters)

FIR DT filters are well-suited to getting linear phase and are therefore very widely used.