BINGHAMTON UNIVERSITY
State University of New York

EECE 301
Signals & Systems

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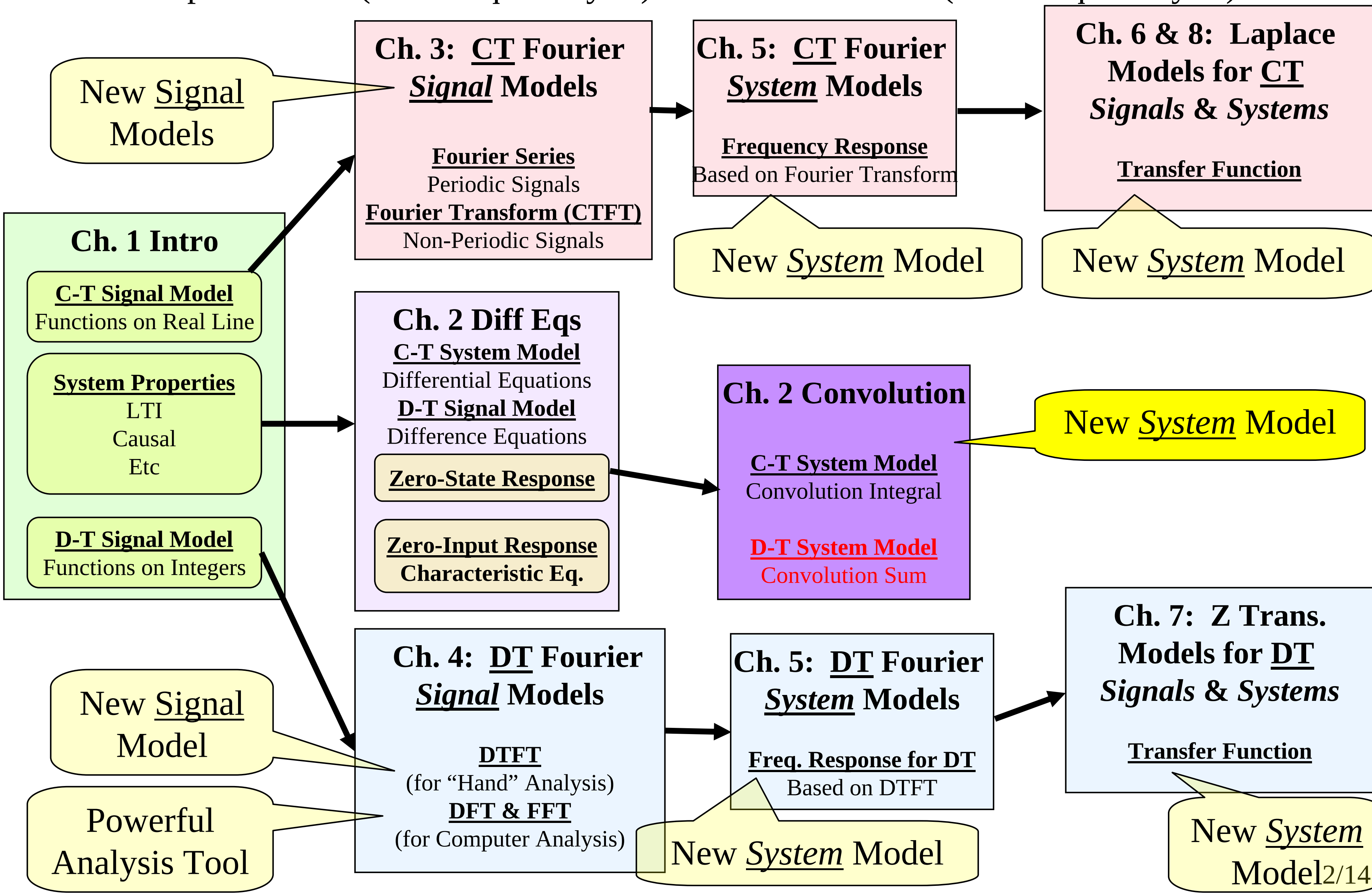
Note Set #8

- D-T Convolution: The Tool for Finding the Zero-State Response
- Reading Assignment: Section 2.1-2.2 of Kamen and Heck

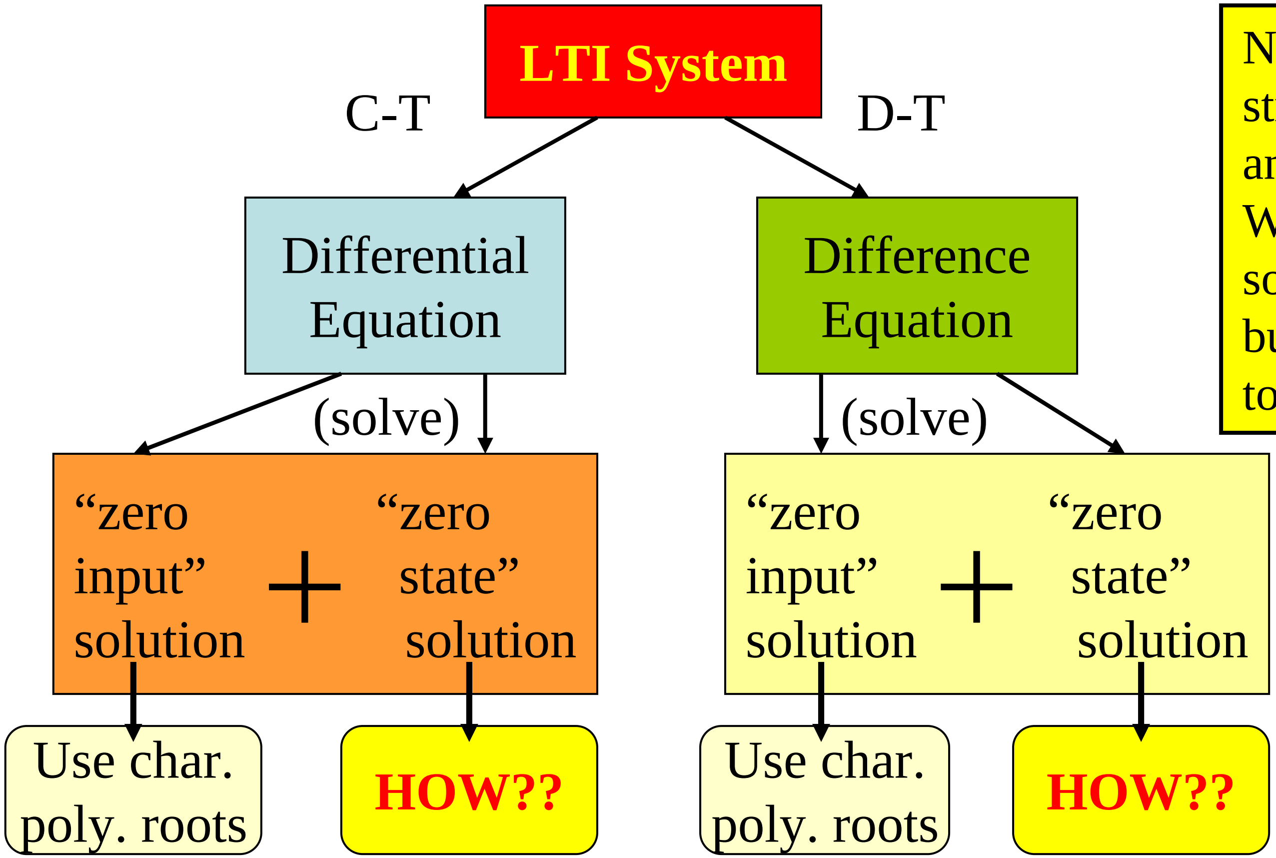
Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



Convolution

Our Interest: **Finding the output of LTI systems (D-T & C-T cases)**



Notice the parallel structure between C-T and D-T systems! We'll see that they are solved using similar but slightly different tools!!!

Our focus in this chapter will be on finding the zero-state solution… (we already know how to find the zero-input solution for C-T differential equations and later we'll learn how to do that for D-T difference equations)

How do we find the Zero-State Response?

(Remember… that is the response (i.e., output) of the system to a specific input when the system has zero initial conditions)

Recall that in the examples for <u>differential</u> equations we always saw:

$$y_{ZS}(t) = \int_{t_0}^{t} h(t-\lambda)x(\lambda)d\lambda$$

C-T "convolution"

Where does this come from?

How do we deal with it?

Recall that in the examples for <u>difference</u> equations we saw:

$$y_{ZS}[n] = \sum_{i=1}^{n} h[n-i]x[i]$$

D-T "convolution"

Where does this come from?

How do we deal with it?

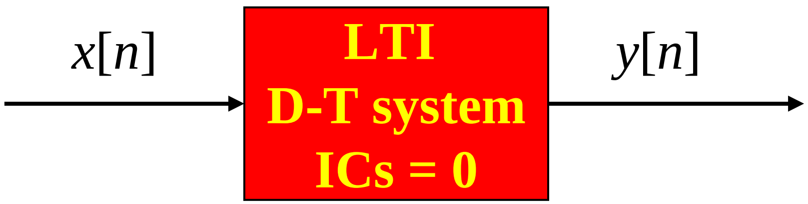
We'll handle D-T systems first because they are easier to understand!

Convolution for LTI D-T systems

We are trying to find $y_{ZS}(t)$… so the ICs = 0

i.e. no stored “energy”

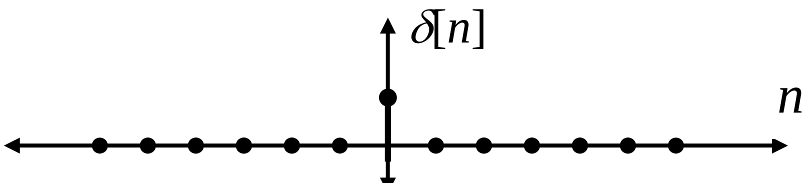
We’ll drop the “zs” subscript to make the notation easier!

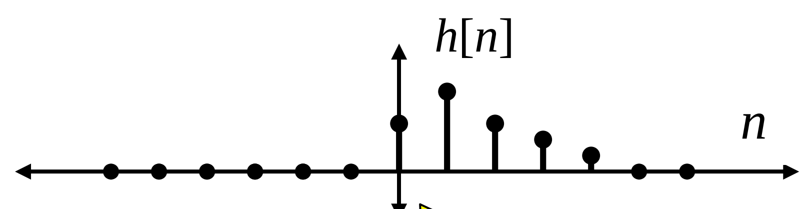


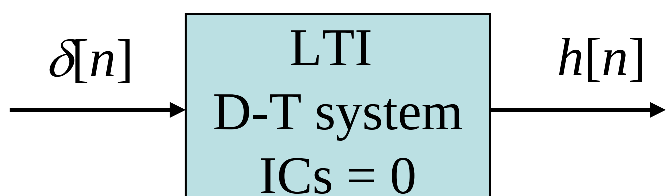
Before we can find the Z-S outptut… we need something first:

Impulse Response (Warning: book calls it “unit-pulse response”)

The impulse response $h[n]$ is what “comes out” when $\delta[n]$ “goes in” w/ ICs=0



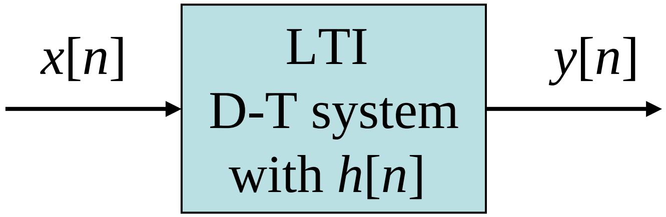




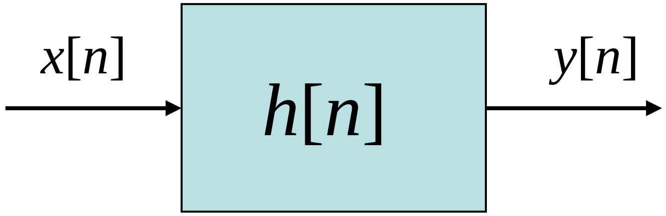
Note: If system is causal, then $h[n] = 0$ for $n < 0$

The impulse response $h[n]$ uniquely describes the system… so we can identify the system by specifying its impulse response $h[n]$.

Thus, we often show the system using a block diagram with the system's impulse response $h[n]$ inside the box representing the system:



Because impulse response is only defined for LTI systems, if you see a box with the symbol $h[n]$ inside it you can assume that the system is an LTI system.



How do we know/get the impulse response $h[n]$?

Many possible ways:

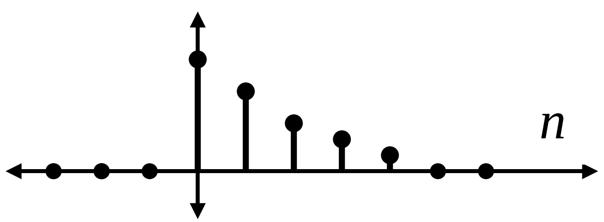
1. Given by the designer of D-T systems
2. Measured experimentally
   - -Put in sequence . . . 0 0 1 0 0 0 . . .
   - -See what comes out
3. Mathematically analyze the D-T system
   - -Given difference equation
   - -Derive $h[n]$

There are many ways to do this, as we will see!

In what form will we know $h[n]$?

1. $h[n]$ known analytically as a function

Ex: $$h[n] = \left(\frac{1}{2}\right)^n u[n]$$



2. $h[n]$ known numerically as a finite-duration sequence

Ex:

n	0	1	2	3	4	5	
$h[n]$	0.5	1	2.1	1.3	.6	0	. . .

We assume that $h[n] = 0$ for $n < 0$

Example of analytically finding $h[n]$

Given a system described by a 1st order difference equation:

$$y[n] = -ay[n-1] + bx[n] \qquad (a \text{ and } b \text{ are arbitrary numbers})$$

Recall that $h[n]$ is what comes out when $\delta[n]$ goes in (with zero ICs).
So we can re-write the above difference equation as follows:

$$h[n] = -ah[n-1] + b\delta[n]$$

Here we solve for $h[n]$ recursively and then examine the results to deduce a closed-form solution (note: we can't always use this "deductive" approach):

n	$\delta[n]$	$h[n]$
-1	0	$-a\times 0 + b\times 0 = 0$
0	1	$-a\times 0 + b\times 1 = b$
1	0	$-ab + b\times 0 = -ab$
2	0	$-a\times(-ab) = (-a)^2 b$
3	0	$-a\times(-a)^2 b = (-a)^3 b$

By examining these results we see…

$$= b(-a)^n u[n]$$

$$h[n] = b(-a)^n u[n]$$

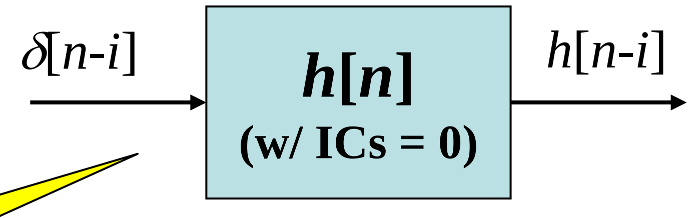
So… we now have the impulse response for this system!!! Next we'll learn how to <u>use</u> it to <u>solve</u> for the <u>zero-state response</u>!!!

Q: How do we use $h[n]$ to find the Zero-State Response?

A: "Convolution" We'll go through three analysis steps that will derive "The General Answer" that convolution is what we need to do to find the zero-state response

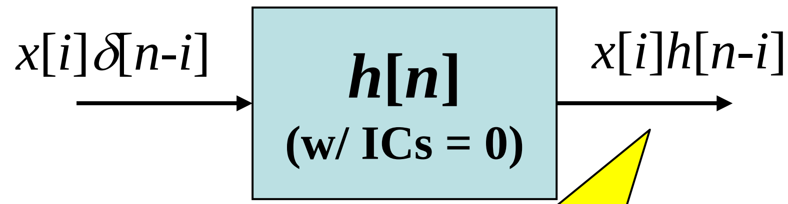
After that… we won't need to re-do these steps… **we'll just "Do Convolution"**

Step 1: Using time-invariance we know:



Shifted input gives shifted output

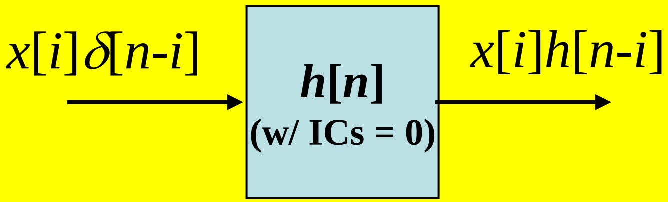
Step 2: Use "homogeneity" part of linearity:

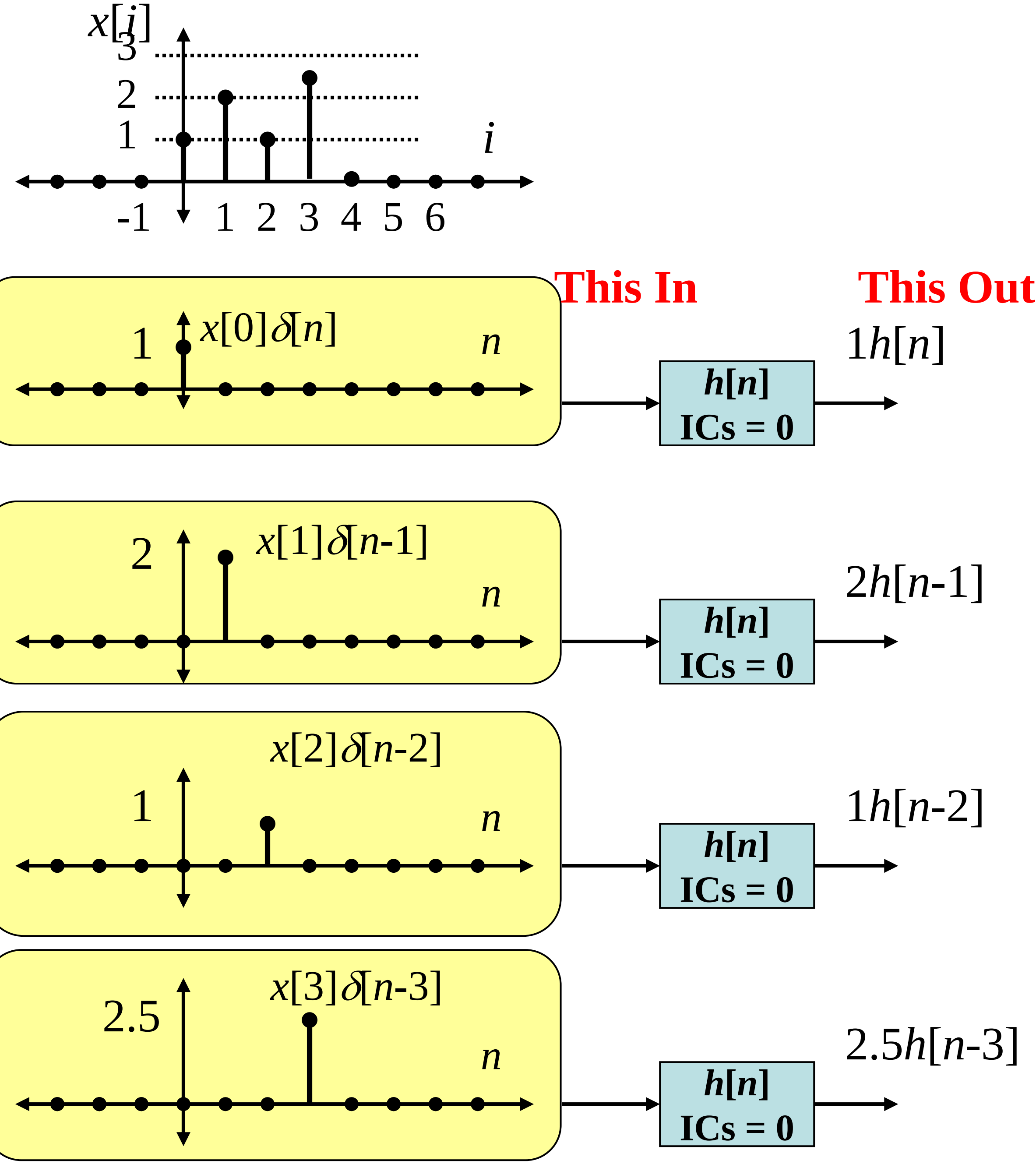


The input is a function of n so we view $x[i]$ as a fixed number for a given i

So… we scale the output by the same fixed number

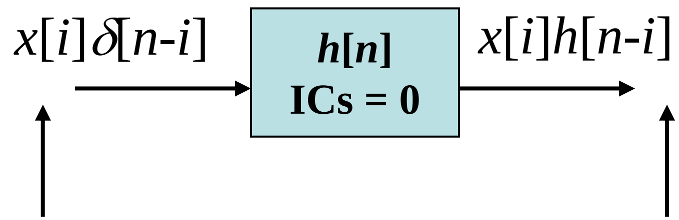
Let's <u>see</u> step 2…
for a specific input:





Step 3: Use "additivity" part of linearity

In Step 2 we looked at inputs like this:



For each i, a different input $\Rightarrow$ For each i, a different response

Now we use the additivity part of linearity:

Put the Sum of Those Inputs In $\Rightarrow$ Get the Sum of Their Responses Out

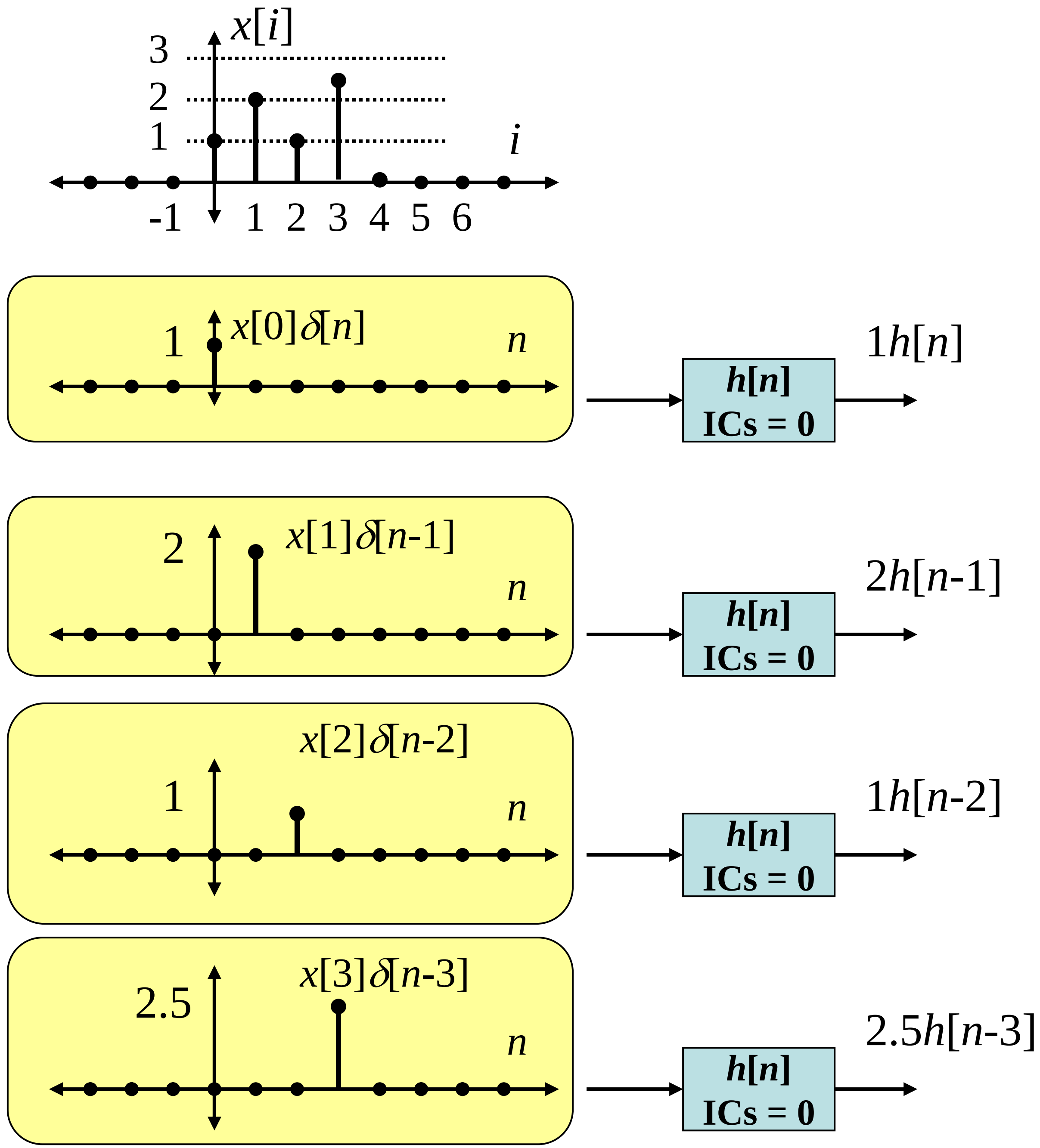
Input: $\sum_{i=-\infty}^{\infty} x[i]\delta[n-i]$ $\Rightarrow$ **Output:** $\sum_{i=-\infty}^{\infty} x[i]h[n-i]$

But… what is this??
On the next slide we show that it is
the desired input signal $x[n]$!

Let's <u>see</u> step 3 for a specific input:

$$\sum_{i=-\infty}^{\infty} x[i]\delta[n-i]$$

<u>Note:</u> The Sum of these "x-weighted" impulses gives $x[n]$!!

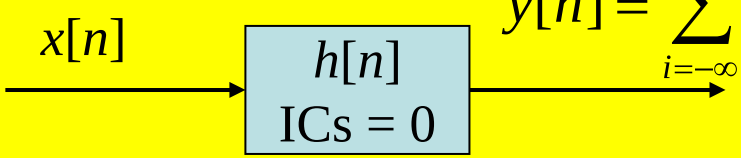


So… what we've seen is this:

Input: $\sum_{i=-\infty}^{\infty} x[i]\delta[n-i]$ → **Output:** $\sum_{i=-\infty}^{\infty} x[i]h[n-i]$

$$\underbrace{\sum_{i=-\infty}^{\infty} x[i]\delta[n-i]}_{= x[n]}$$

Or in other words… we've derived an expression that tells what comes out of a D-T LTI system with input *x*[*n*]:



$$y[n] = \sum_{i=-\infty}^{\infty} x[i]h[n-i]$$ CONVOLUTION!

$$y[n] = x[n] * h[n]$$

Notation for Convolution

So… now that we have derived this result we don't have to do these three steps… we "just" use this equation to find the zero-state output:

$$y[n] = \sum_{i=-\infty}^{\infty} x[i]h[n-i]$$ CONVOLUTION!

Big Picture

For a LTI D-T system in zero state we no longer need the difference equation model…

-Instead we need the impulse response $h[n]$ & convolution

New alternative model!