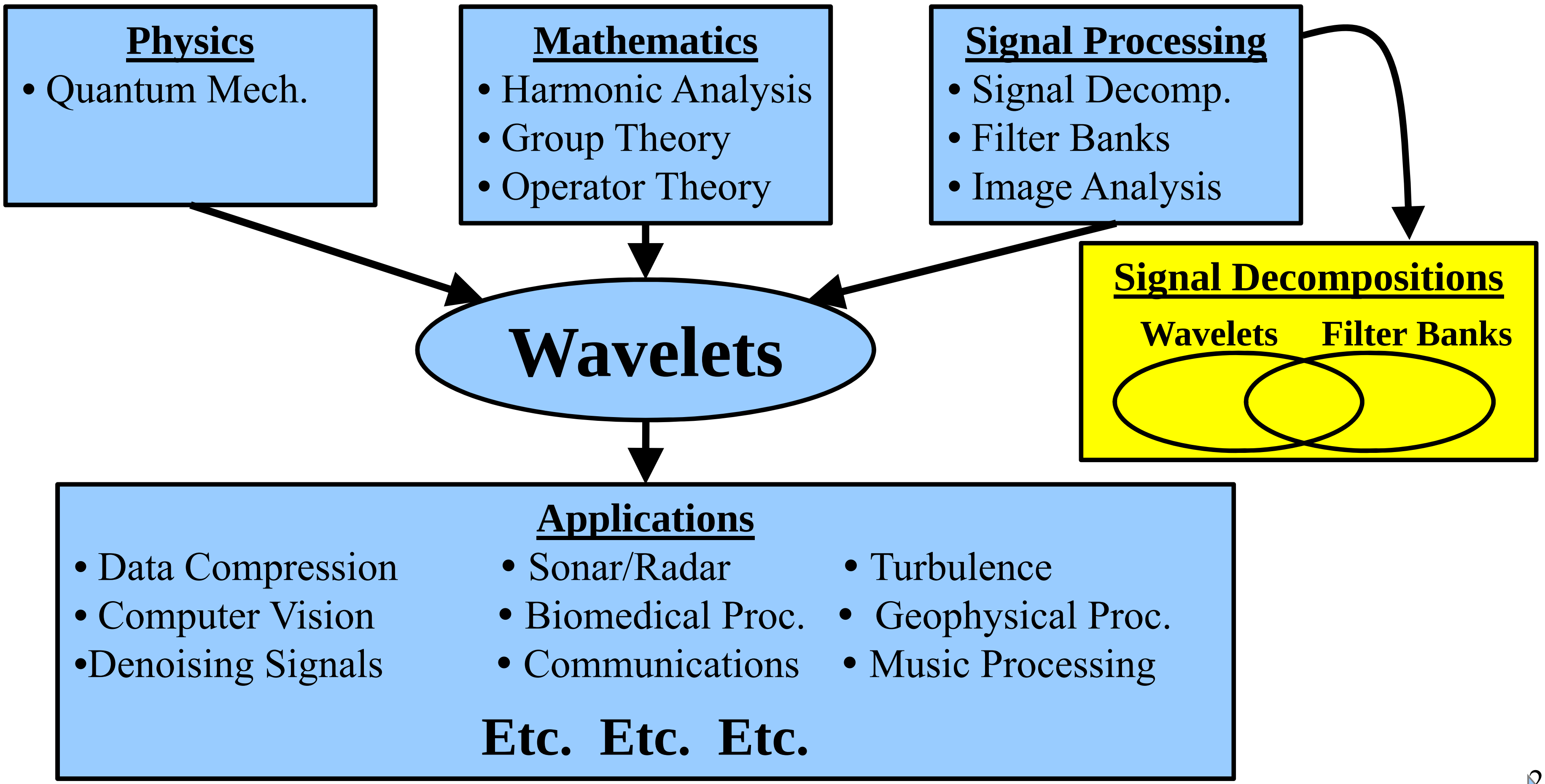
Ch. 15 Wavelet-Based Compression

Origins and Applications

The Wavelet Transform (WT) is a signal processing tool that is replacing the Fourier Transform (FT) in many (but not all!) applications.

WT theory has its origins in ideas in three main areas and now is being applied in countless different areas of application.



So, What's Wrong With The FT?

First, recall the FT:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

Weight @ *f*

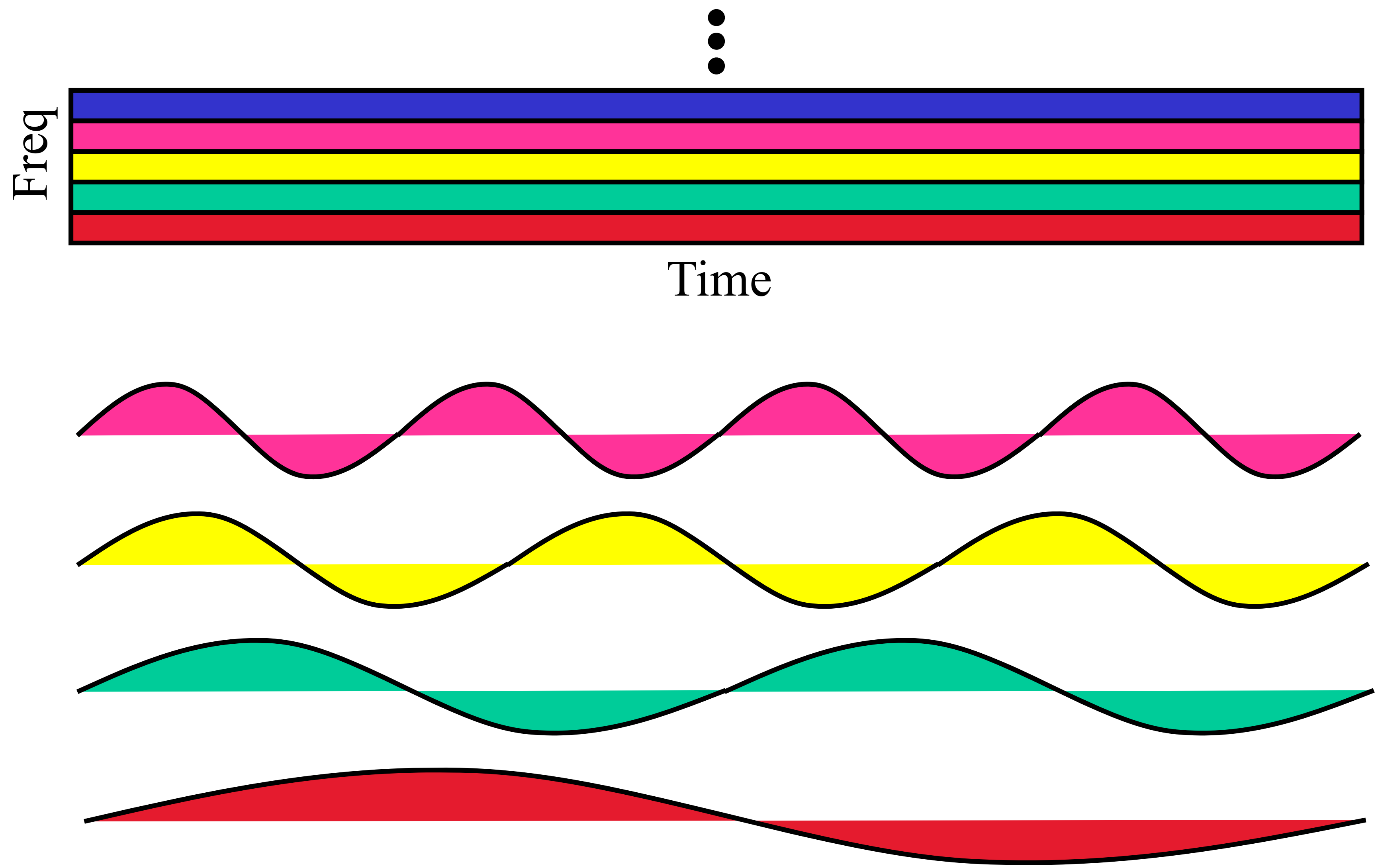
Component @ *f*

Remember: An integral is like a summation… So, the second equation says that we are decomposing *x*(*t*) into a weighted "sum" of complex exponentials (sinusoids!)… The first equation tells what each weight should be.

Note: These components exist for ***ALL*** time!!!

This is not necessarily a good model for real-life signals.

DFT Basis Functions… and "Time-Freq Tiles"



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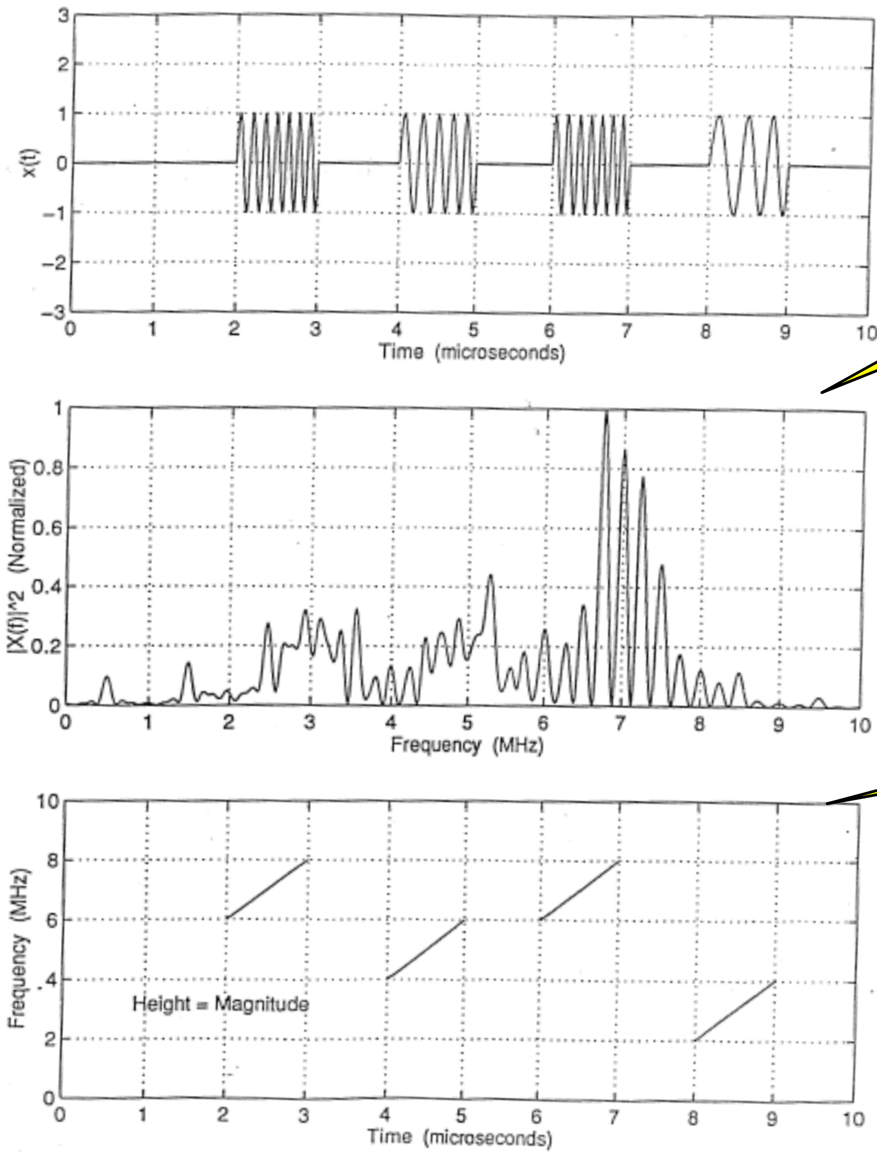
Example: Frequency-Hopping Chirped Pulses



DFT tries to build from freq components that last 10 μsec

The DFT representation is "correct" but does not show us a "joint representation" in time and frequency

This would be a more desirable time-frequency representation

It would show at each time exactly what frequencies were in existence at that time!

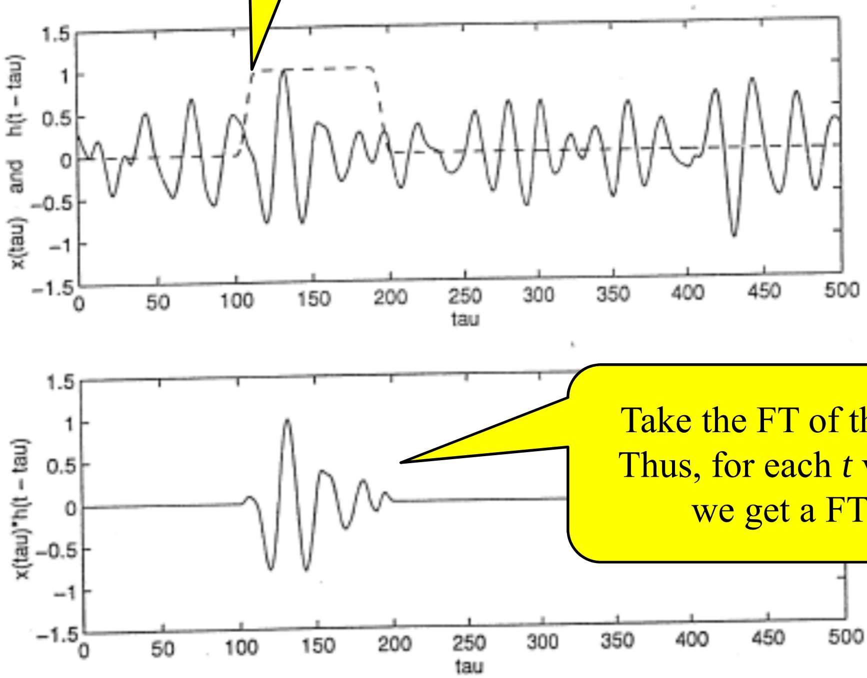
However… such a "perfect" t-f representation is not possible… Heisenberg Uncertainty Principle!

But What About The Short-Time FT (STFT)?

$$X(f,t) = \int_{-\infty}^{\infty} x(\tau) h(\tau - t) e^{-j2\pi f \tau} d\tau$$

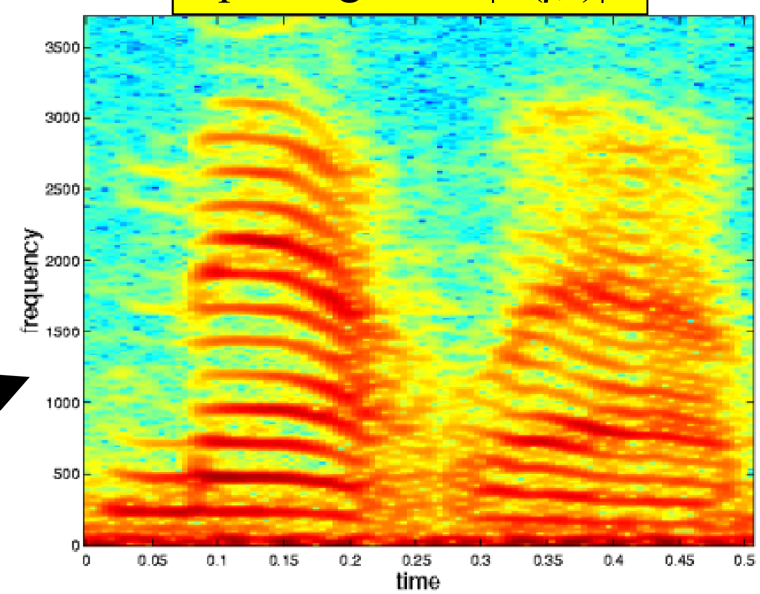
h(τ - *t*)

Dummy Variable!



Take the FT of this…
Thus, for each *t* value we get a FT

Spectrogram = $|X(f,t)|^2$



Selesnick, Ivan, "Short Time Fourier Transform," Connexions, August 9, 2005. http://cnx.org/content/m10570/2.4/.

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STFT T-F Resolution

- The window function $h(t)$ sets the characteristic of how the STFT is able to "probe" the signal $x(t)$.
  - The narrower $h(t)$ is, the better you can resolve the time of occurrence of a feature
  - However… the narrower $h(t)$ is, the wider $H(f)$ is… and that means a reduction in the ability to resolve frequency occurance
    - Just like windowing of the DFT that you've probably studied!
- Each given $h(t)$ has a given time and frequency resolution
  - Δt describes the time resolution
  - Δf describes the frequency resolution
- The Heisenberg Uncertainty Principle states that
  - $$(\Delta t)(\Delta f) \geq \frac{1}{4\pi}$$
  - Improving Time Resolution…. Degrades Frequency Resolution
    - And vice versa

Illustration of Time-Frequency Resolution Trade-Off

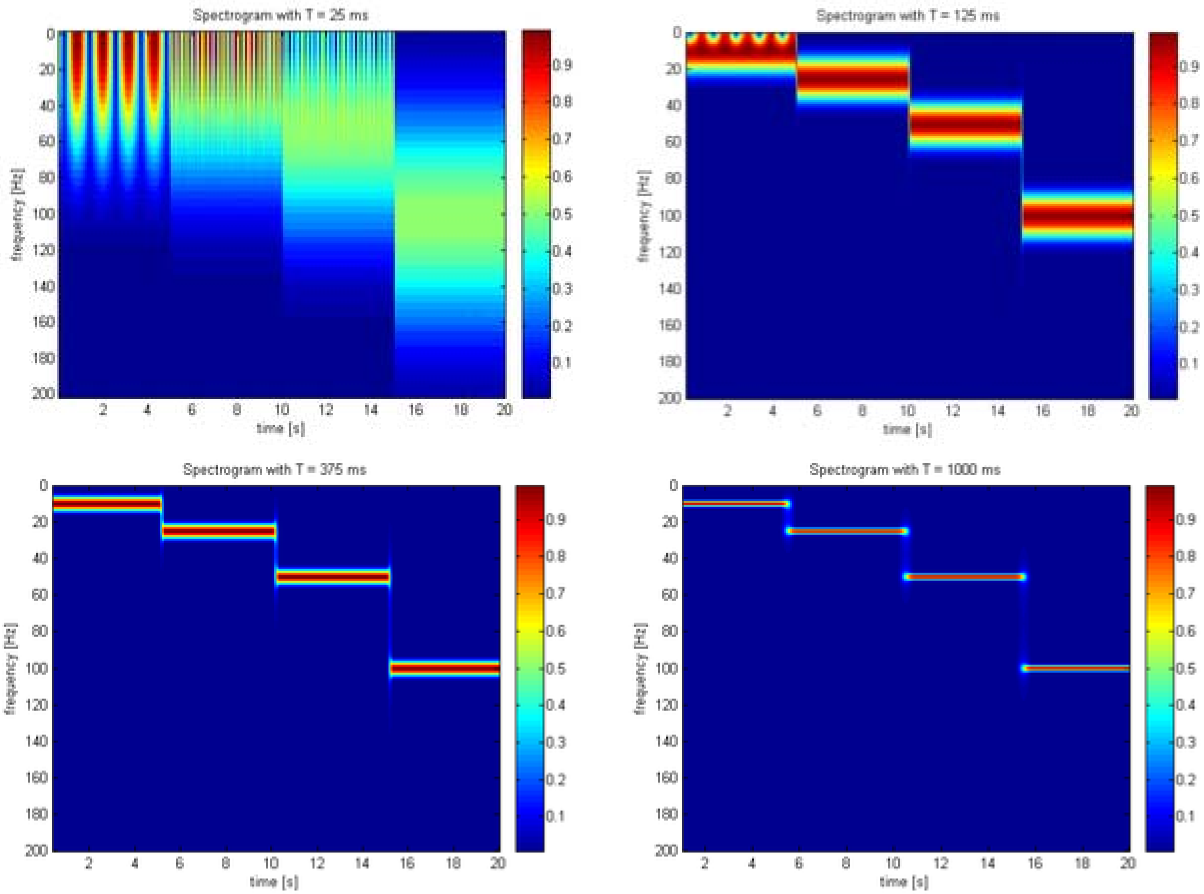
$$x(t) = \begin{cases} \cos(2\pi 10t) & 0 \le t < 5 \\ \cos(2\pi 25t) & 5 \le t < 10 \\ \cos(2\pi 50t) & 10 \le t < 15 \\ \cos(2\pi 100t) & 15 \le t < 20 \end{cases}$$

Window Width 25 ms

Window Width 125 ms

Window Width 375 ms

Window Width 1000 ms



http://en.wikipedia.org/wiki/Short-time_Fourier_transform

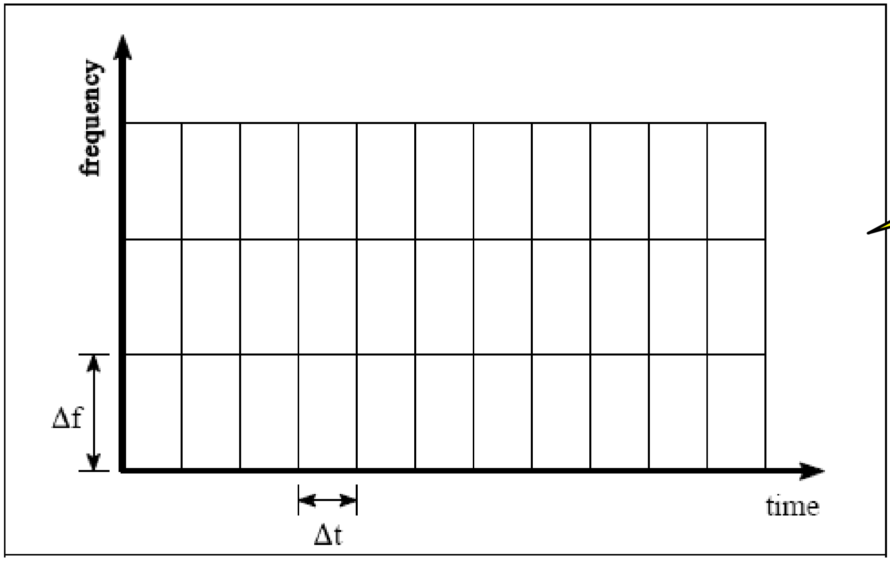
STFT View of Tiling the T-F Plane

Generally only compute the STFT for discrete values of *t* and *f*

$$X(f_m, t_n) = \int_{-\infty}^{\infty} x(\tau)h(\tau - nT)e^{-j2\pi(mF)\tau}d\tau$$

In some applications it is desirable to minimize the number of points in $X(f_m,t_n)$ and that means making T and F as large as possible… $T \approx \Delta t$ and $F \approx \Delta f$

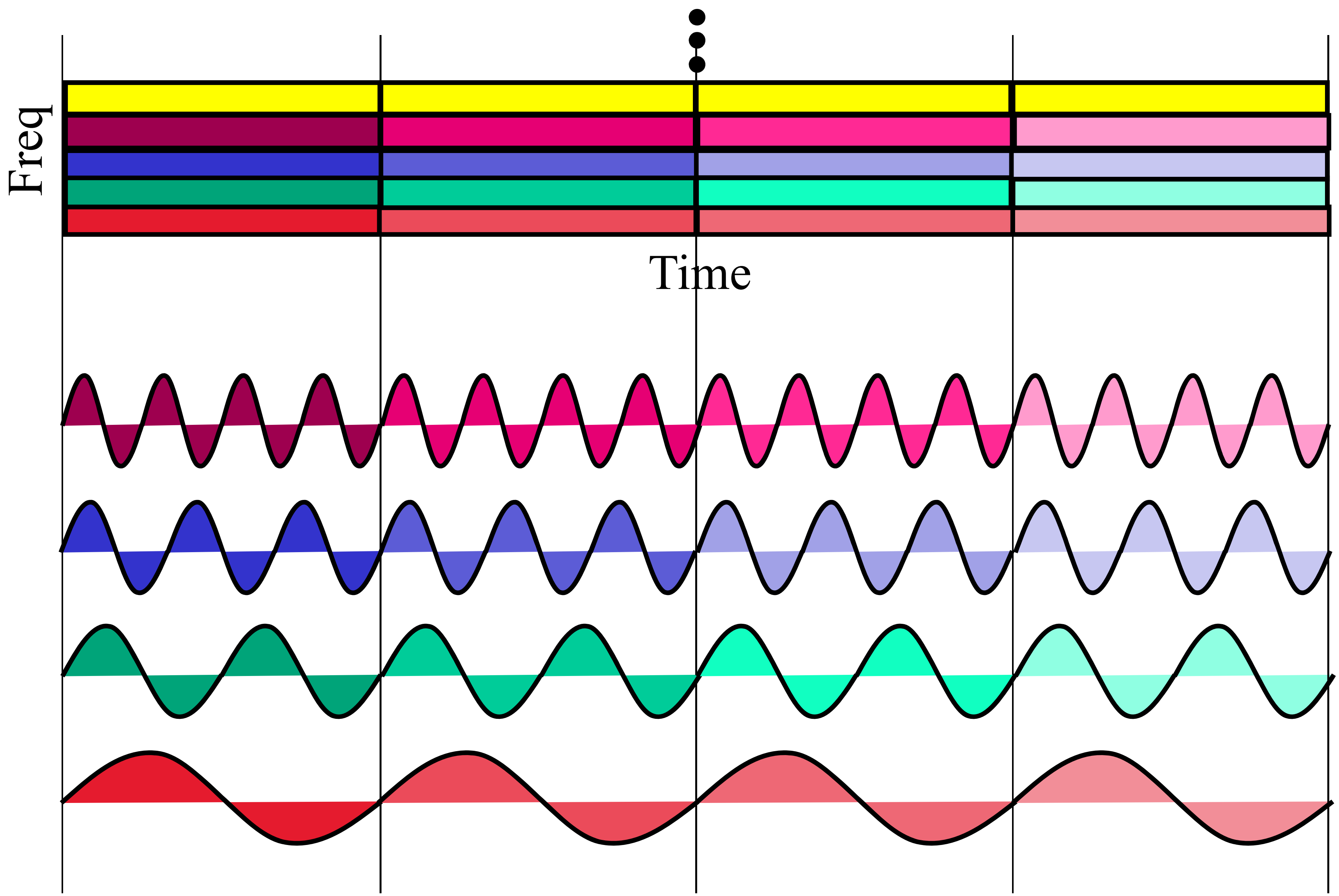
Then each $X(f_m,t_n)$ represents the "content" of the signal in a rectangular cell of dimension Δt by Δf



STFT Tiling of the T-F Plane

STFT tiling consists of a uniform tiling by fixed rectangles

STFT Basis Functions… and "Time-Freq Tiles"



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STFT Disadvantages and Advantages

- The fact that the STFT tiles the plane with cells having the same Δt and Δf is a disadvantage in many application
  - Especially in the data compression!
- This characteristic leads to the following:
  - If you try to make the STFT be a "non-redundant" decomposition (e.g., ON… like is good for data compression…
  - You necessarily get very poor time-frequency resolution
- This is one of the main ways that the WT can help
  - It can provide ON decompositions while still giving good t-f resolution
- However, in applications that do not need a non-redundant decomposition the STFT is still VERY useful and popular
  - Good for applications where humans want to view results of t-f decomposition

So… What *IS* the WT???

Recall the STFT: $$X(f,t) = \int_{-\infty}^{\infty} x(\tau)\underbrace{h(\tau - t)e^{-j2\pi f\tau}}_{\textbf{Basis Functions}}d\tau$$

So… $X(f,t)$ is computing by “comparing” $x(t)$ to each of these basis functions

For the STFT the basis function are created by applying
Time Shift and Frequency Shift to prototype $h(t)$

This leads to the “uniform tiling” we saw before…

And it also causes the problems with the non-redundant form of the STFT

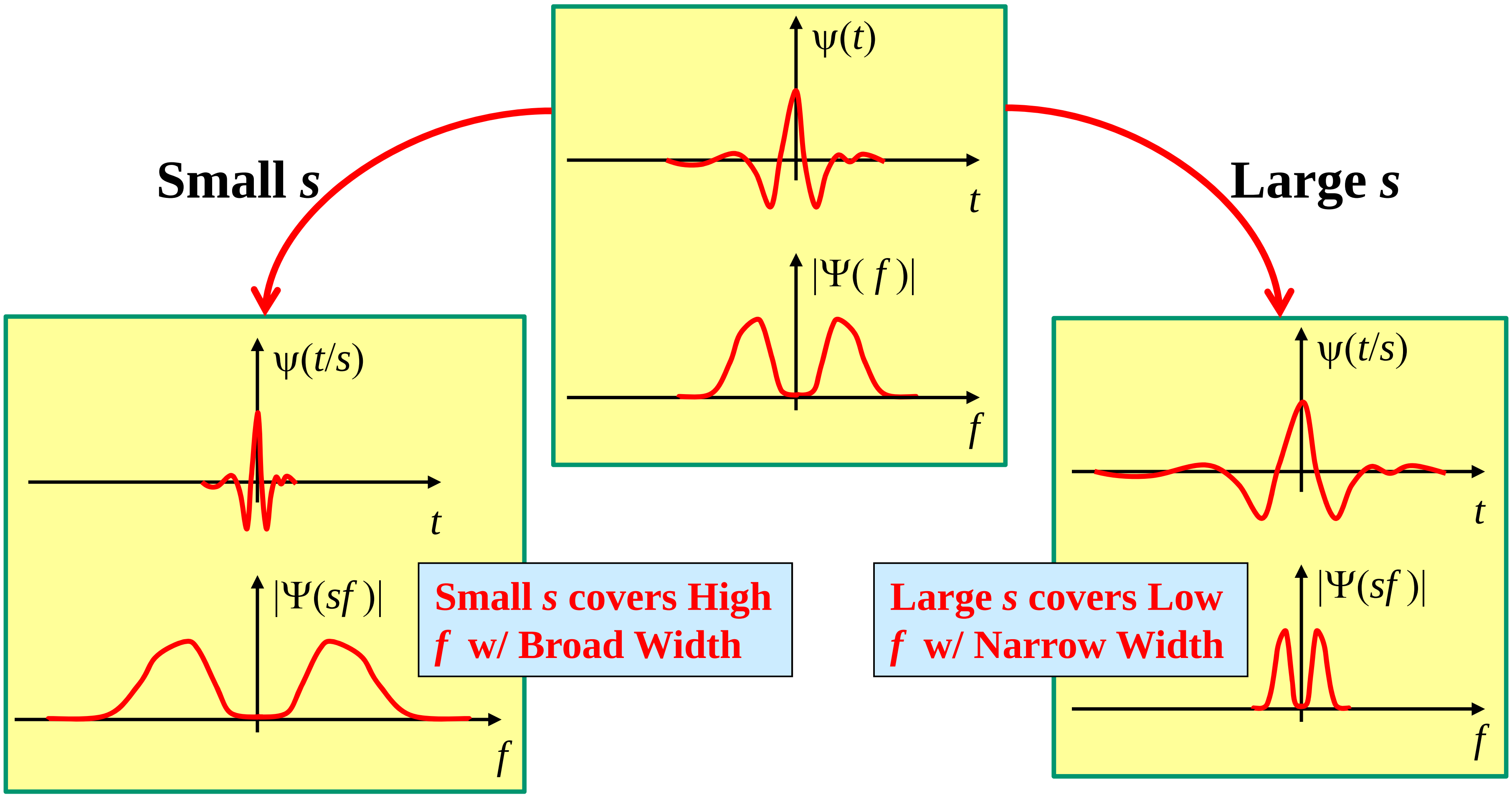
So… we need to find a new way to make T-F basis functions that don’t have these problems!!!

The WT comes about from replacing frequency shifting by time scaling…

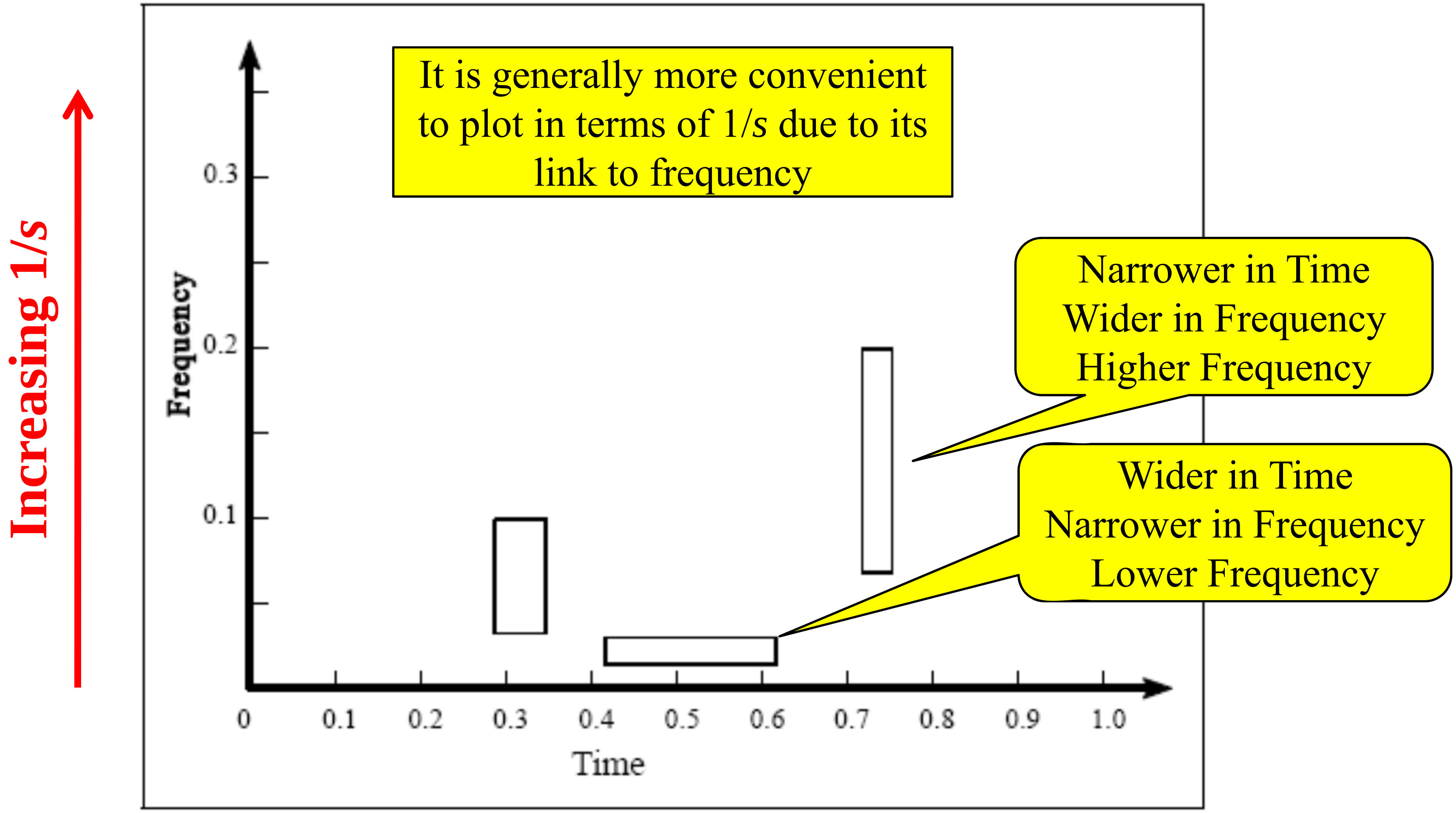
Start with a prototype signal $\psi(t)$ and time scale it:

$$\psi(t/s) \quad \leftrightarrow \quad s\Psi(sf)$$

<u>Increasing *s*</u>: Stretches the signal “Scrunches” the spectrum	**<u>Decreasing *s*</u>**: “Scrunches” the signal Stretches the spectrum



This shows some typical t-f cells for wavelets



We still need to satisfy the uncertainty principle: $(\Delta t)(\Delta f) \geq \dfrac{1}{4\pi}$

But now Δt and Δf are adjusted depending on what region of frequency is being "probed".

All this leads to… <u>**The Wavelet Transform**</u>**:**

$$X(s,t) = \int_{-\infty}^{\infty} x(\tau)\left[\tfrac{1}{\sqrt{s}}\psi\left(\frac{\tau - t}{s}\right)\right] d\tau, \quad s > 0$$

ψ(*t*) is called the Mother Wavelet

<u>The Inverse Wavelet Transform</u> (Reconstruction Formula):

$$x(t) = \frac{1}{C_\psi}\int_0^{\infty}\int_{-\infty}^{\infty} X(s,\tau)\left[\tfrac{1}{\sqrt{s}}\psi\left(\frac{\tau - t}{s}\right)\right]\frac{ds\,d\tau}{s^2},$$

$$C_\psi = \int_0^{\infty}\frac{|\Psi(\omega)|^2}{\omega} d\omega$$

<u>Requirements for a Mother Wavelet are:</u>

Finite Energy: $\psi(t) \in L^2(\mathbb{R}) \;\Rightarrow\; \int_{-\infty}^{\infty}\psi^2(t)dt < \infty \underbrace{\Rightarrow}_{Parseval} \int_{-\infty}^{\infty}|\Psi(\omega)|^2 d\omega < \infty$

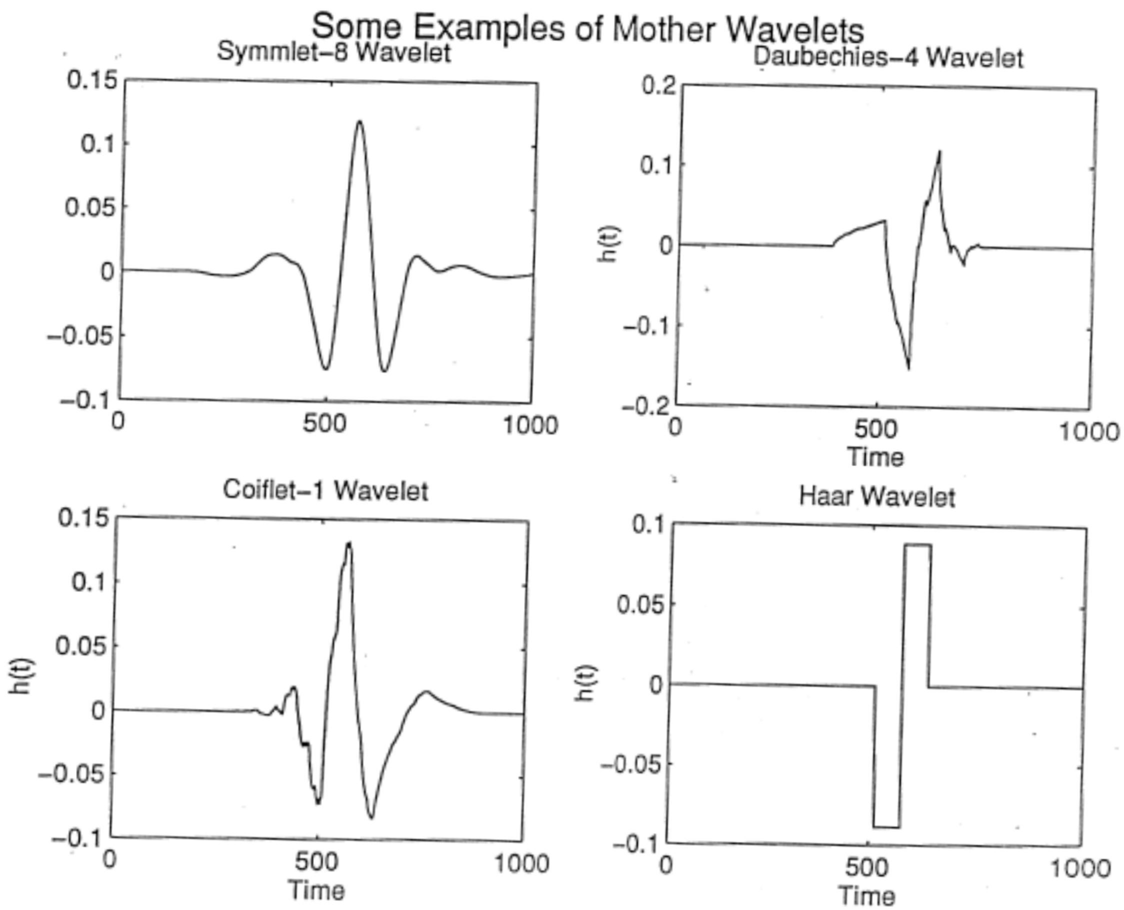
Admissibility Condition: $\int_0^{\infty}\frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty$

- $|\Psi(\omega)|^2$ must go to zero fast enough as $\omega \to 0$
- $|\Psi(\omega)|^2$ must go to zero fast enough as $\omega \to \infty$

ψ(*t*) must be a bandpass signal

The prototype basis function $\psi(t)$ is called the mother wavelet…
All the other basis functions come from scaling and shifting the mother wavelet.
There are many choices of mother wavelet:

- Each gives rise to a slightly different WT
- …with slightly different characteristics
- …suited to different applications.



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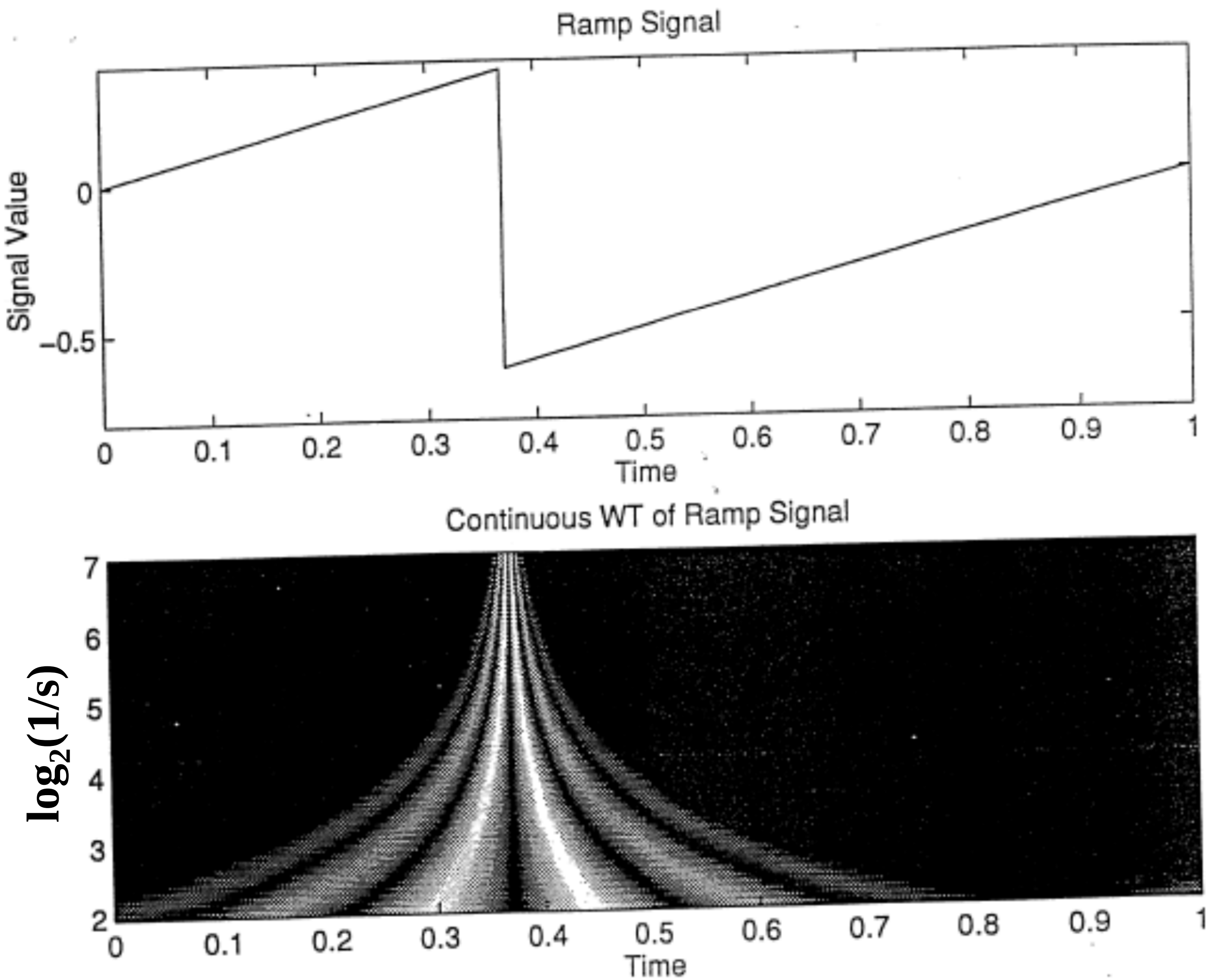
Example of a WT



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Non-Redundant Form of WT

It is often desirable to use a discrete form of the WT that is "non-redundant"… that is, we only need $X(s,t)$ on a discrete set of s and t values to reconstruct $x(t)$.

Under some conditions it is possible to do this with only s and t taking these values:

$$s = 2^m \quad t = n2^m \quad \text{for} \quad m = \ldots -3,-2,-1,0,1,2,3,\ldots$$
$$n = \ldots -3,-2,-1,0,1,2,3,\ldots$$

In practice you truncate the range of m and n

Lower values of m $\Rightarrow$ Smaller values of s $\Rightarrow$ Higher Frequency

Incrementing m doubles the scale value and doubles the time spacing

Then the WT becomes a countably infinite collection of numbers (recall the Fourier series vs. the Fourier transform):

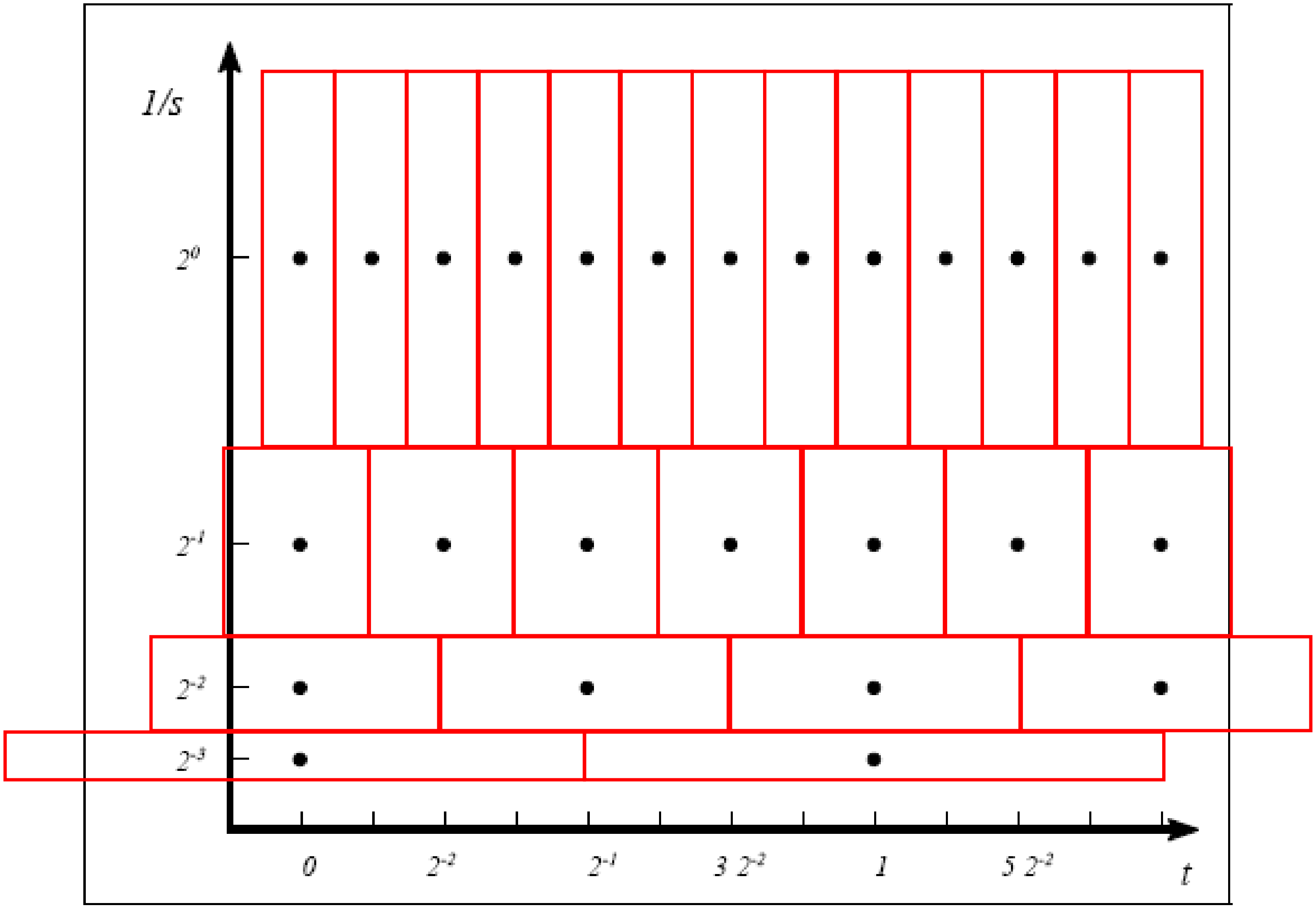
$$X(s,t) = \int_{-\infty}^{\infty} x(\tau)\left[\frac{1}{\sqrt{s}}\psi\left(\frac{\tau - t}{s}\right)\right] d\tau \quad \Rightarrow \quad X_{m,n} = \int_{-\infty}^{\infty} x(\tau)\underbrace{\left[2^{-m/2}\psi\left(2^{-m}\tau - n\right)\right]}_{\triangleq \psi_{m,n}(\tau)} d\tau, \quad m,n \in \mathbb{Z}$$

Advantages of This Form

- The $\psi_{m,n}(t)$ can be an ON basis for L^2
- Good for Data Compression
- Simple, numerically stable inverse
- Leads to efficient discrete-time form

$$x(t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} X_{m,n}\left[2^{-m/2}\psi\left(2^{-m}t - n\right)\right]$$

This leads to the sampling and the tiling of the *t*-1/*s* plane as shown below:



(Time)-(Inverse Scale) Sampling Grid for Wavelet Transform

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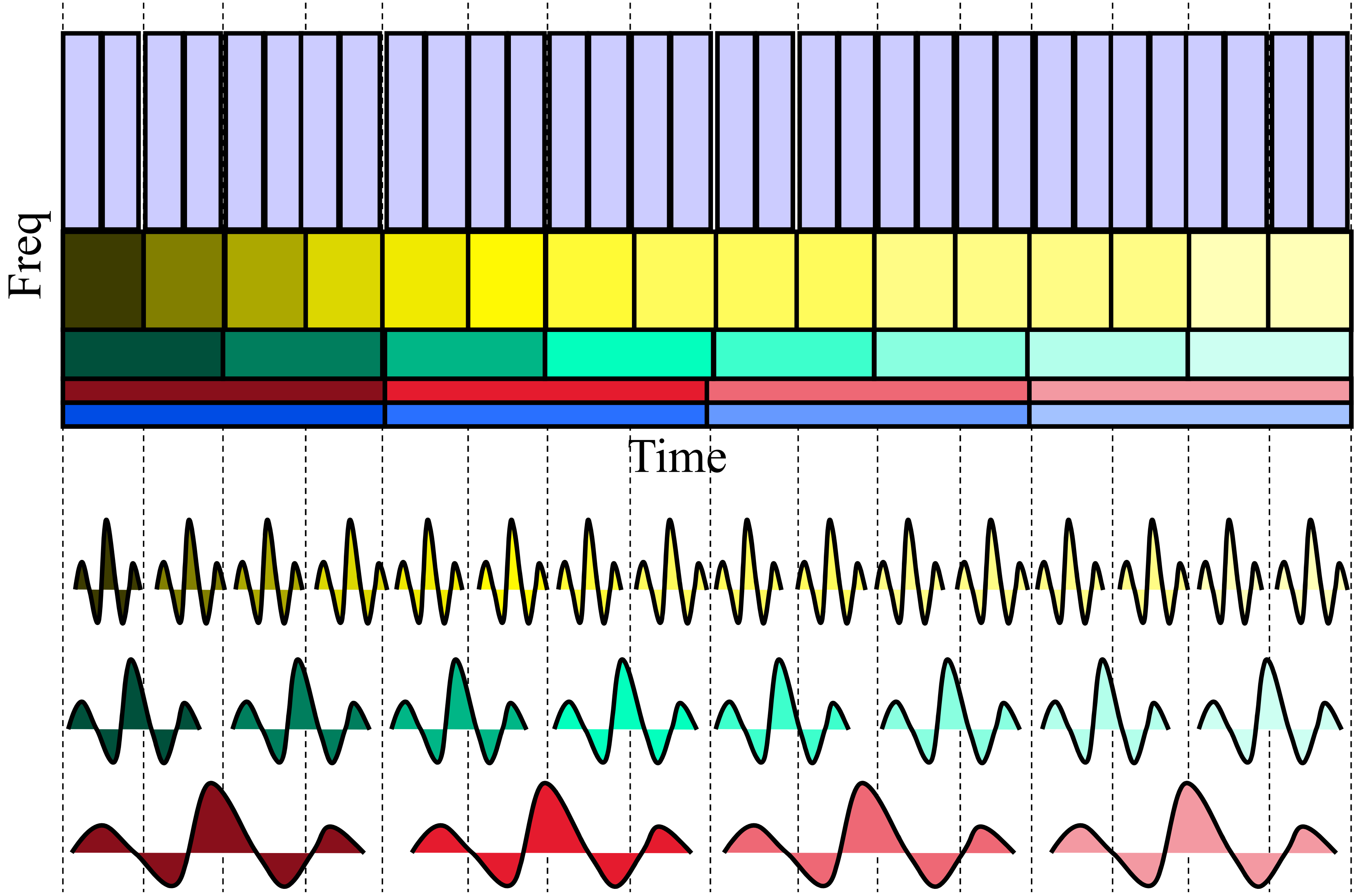
WT Tiling and Basis Functions



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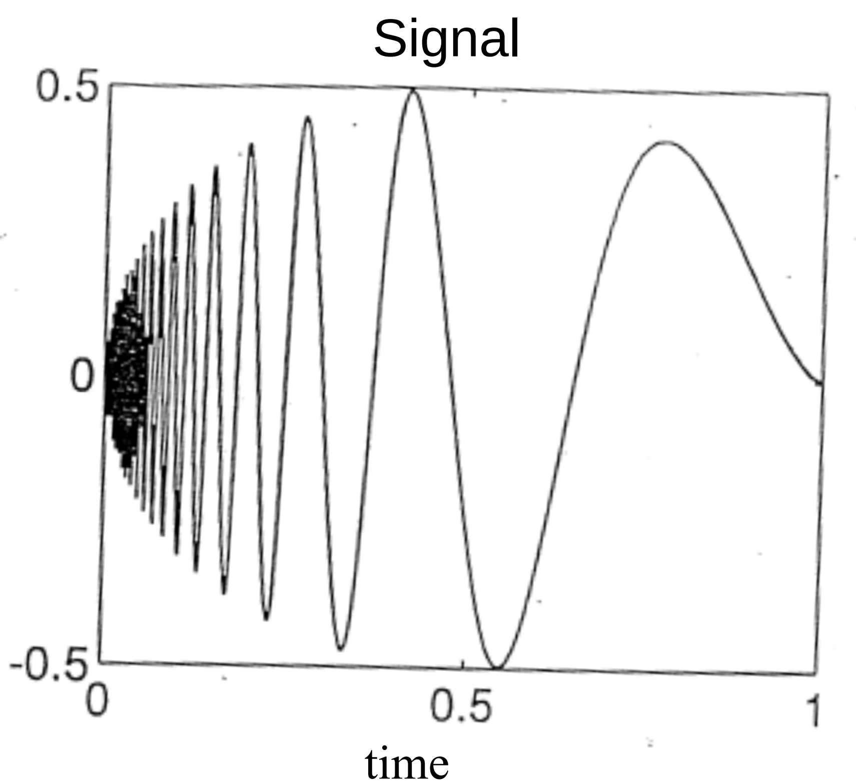
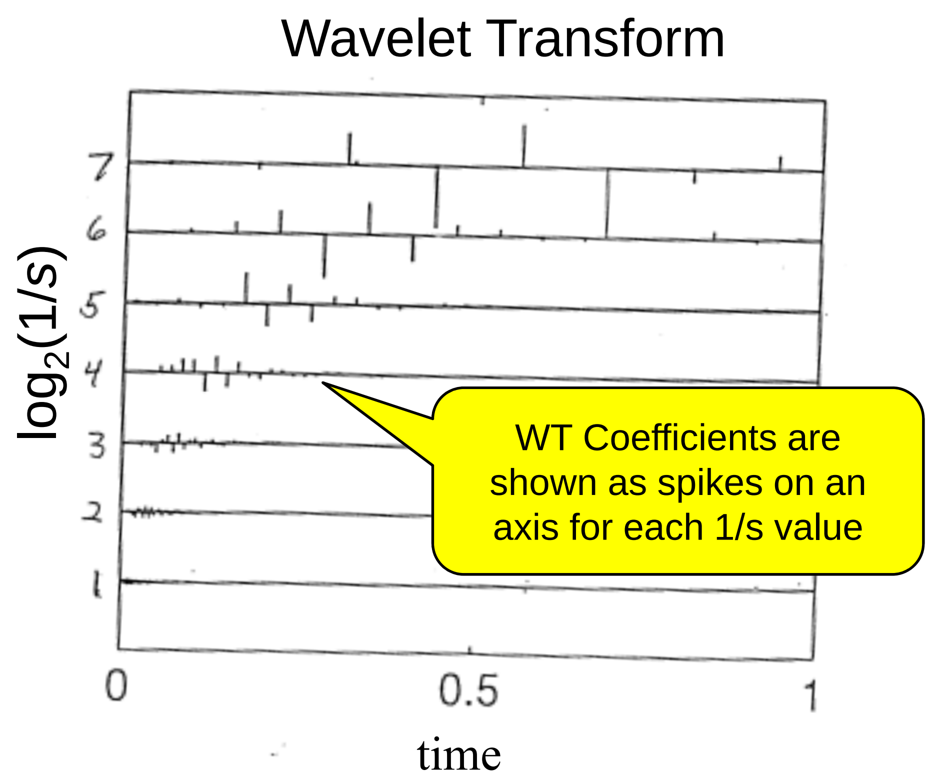
Example #1

- A synthetic Chirp Signal
  - Frequency decreases with time
  - Amplitude increases with time
- Notice that
  - High frequency components dominate early
  - Low frequency components dominate later
  - Low frequency components are stronger





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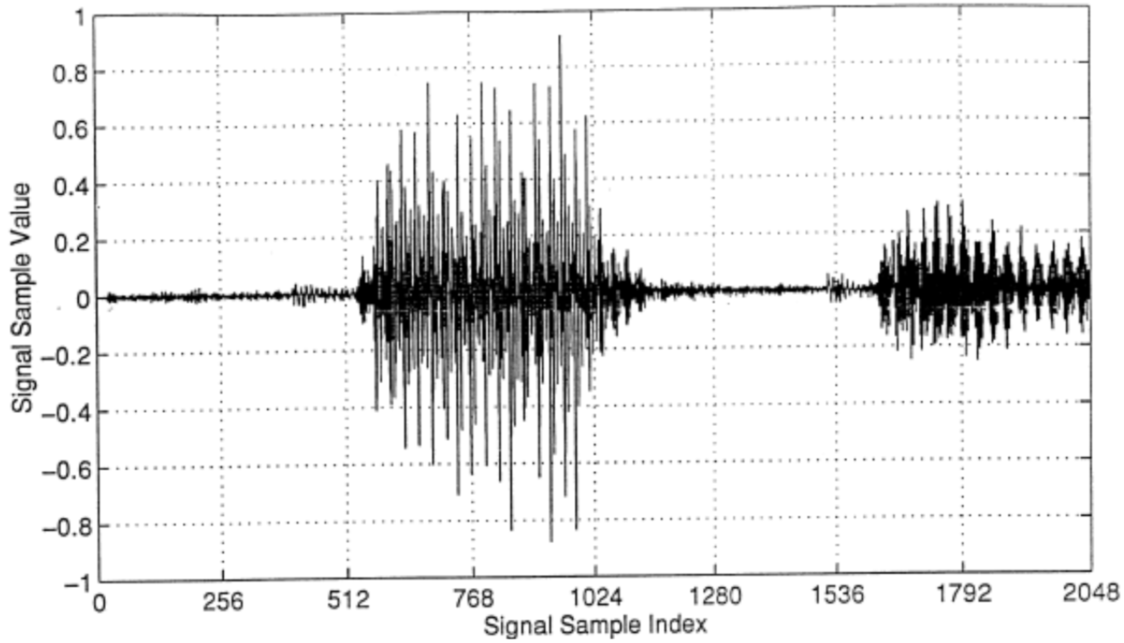
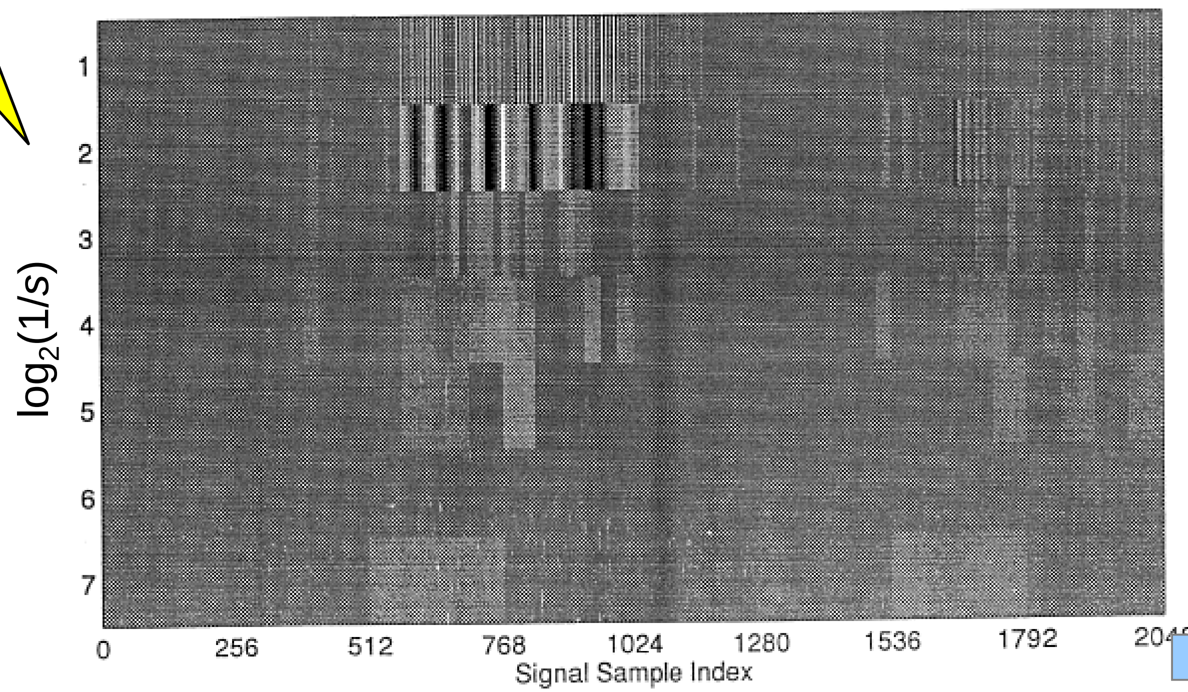
Example #2

Speech Signal



- WT coeff's are displayed as gray- scale blocks
- WT coeffs concentrated
- Blocks closely spaced at high *f*
- Blocks widely spaced at low *f*



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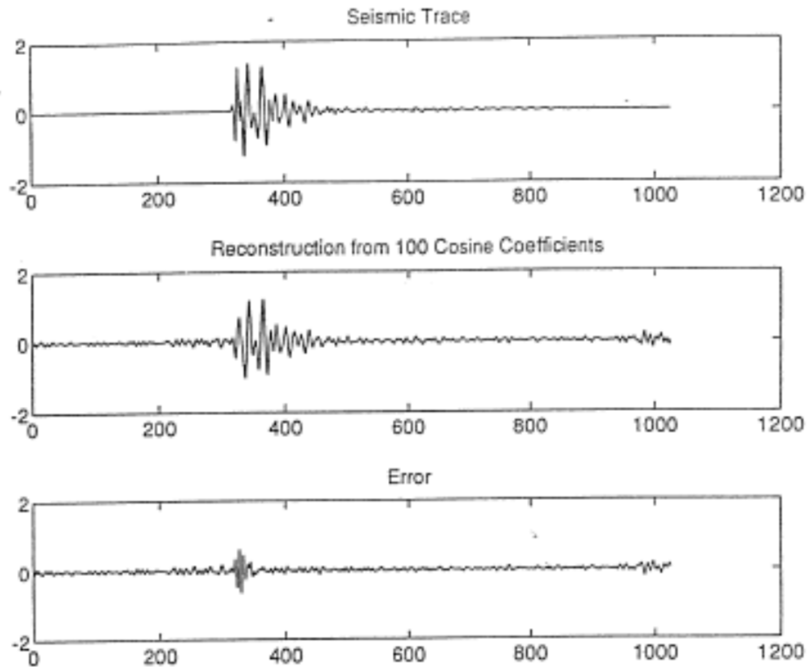
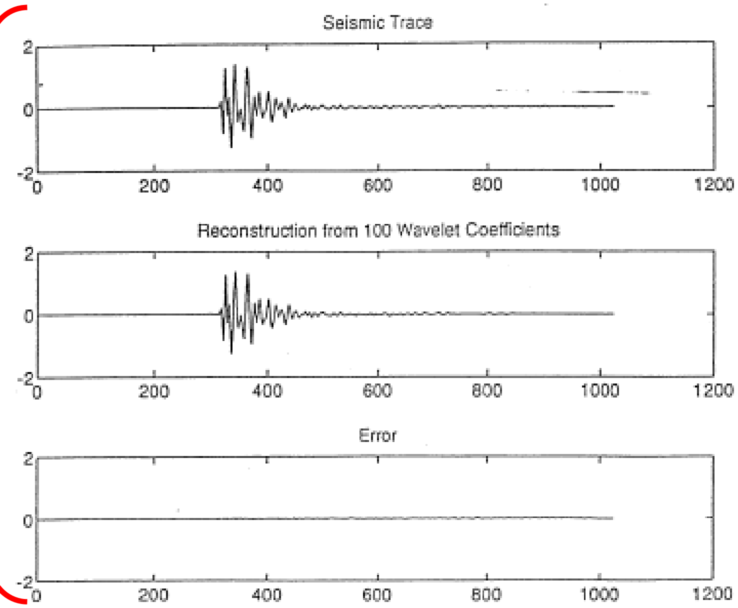
Example #3: Effectiveness of WT T-F Localization Properties



- Seismic Trace Signal
- Keep 100 largest DCT coefficients
- Large Error…
  especially at start of transient



- Same Seismic Trace Signal
- Keep 100 largest WT coefficients
- SMALL Error…
  even at start of transient

Next

Summary So Far: What is a Wavelet Transform?

- Note that there are many ways to decompose a signal. Some are:
  - Fourier series: basis functions are harmonic sinusoids;
  - Fourier transform (FT): basis functions are nonharmonic sinusoids;
  - Walsh decomposition: basis functions are "harmonic" square waves;
  - Karhunen-Loeve decomp: basis functions are eigenfunctions of covariance;
  - Short-Time FT (STFT): basis functions are windowed, nonharmonic sinusoids;
    - Provides a time-frequency viewpoint
  - **Wavelet Transform**: basis functions are time-shifted and time-scaled versions of a mother wavelet
    - Provides a time-scale viewpoint

$$\psi_{m,n}(t) = 2^{-m/2}\psi\left(2^{-m}t - n\right)$$

$$x(t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} X_{m,n}\psi_{m,n}(t)$$

$$X_{m,n} = \int_{-\infty}^{\infty} x(\tau)\psi_{m,n}(\tau)d\tau \,,$$

- Wavelet transform also provides time-frequency view: 1/scale relates to *f*
  - Decomposes signal in terms of duration-limited, band-pass components
    - high-frequency components are short-duration, wide-band
    - low-frequency components are longer-duration, narrow-band
  - Can provide combo of good time-frequency localization and orthogonality
    - the STFT can't do this

Fourier Development vs. Wavelet Development

- Fourier and others:
  - expansion functions are chosen, then properties of transform are found
- Wavelets
  - desired properties are mathematically imposed
  - the needed expansion functions are then derived
- Why are there so many different wavelets?
  - the basic desired property constraints don't use all the degrees of freedom
  - remaining degrees of freedom are used to achieve secondary properties
    - these secondary properties are usually application-specific
    - the primary properties are generally application-nonspecific

Why are Wavelets Effective?

- Provide a good basis for a large signal class
  - wavelet coefficients drop-off rapidly…
  - thus, good for compression, denoising, detection/recognition
  - goal of any expansion is
    - have the coefficients provide more info about signal than time-domain
    - have most of the coefficients be very small (***sparse*** representation)
  - FT is not sparse for transients… WT is sparse for many signals
- Accurate local description and separation of signal characteristics
  - Fourier puts localization info in the phase in a complicated way
  - STFT can't give localization ***and*** orthogonality
- Wavelets can be adjusted or adapted to application
  - remaining degrees of freedom are used to achieve goals
- Computation of wavelet coefficient is well-suited to computer
  - no derivatives or integrals needed
  - turns out to be a digital filter bank… as we will see.

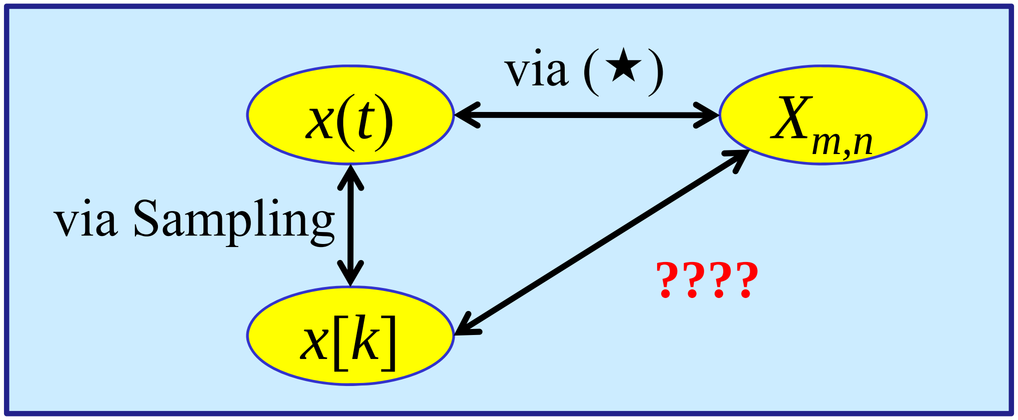
The Discrete-Time WT

Recall the formula for the WT coefficients:

$$X_{m,n} = \int_{-\infty}^{\infty} x(\tau) \underbrace{\left[2^{-m/2} \psi\left(2^{-m} \tau - n \right) \right]}_{\triangleq \psi_{m,n}(\tau)} d\tau, \quad m,n \in \mathbb{Z} \qquad (\bigstar)$$

If the signal $x(t)$ is bandlimited to B Hz, we can represent it by its samples taken every $T_S = 1/2B$ seconds: $x[k] = x(kT_s)$.

Our Goal: Since the samples $x[k]$ uniquely and completely describe $x(t)$, they should also uniquely and completely describe the WT coefficients $X_{m,n}$ … **HOW DO WE DO IT??**



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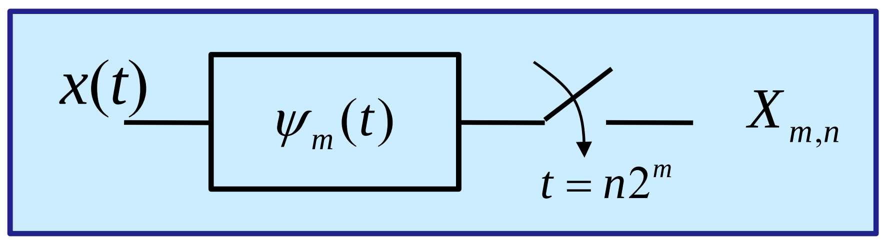
Mathematically Interpret the DWT equation in terms of Signal Processing:

$$X_{m,n} = \int_{-\infty}^{\infty} x(\tau)\left[2^{-m/2}\psi\left(2^{-m}\tau - n\right)\right]d\tau,$$

$$= \int_{-\infty}^{\infty} x(\tau)\underbrace{\left[2^{-m/2}\psi\left(2^{-m}(\tau - n2^{m})\right)\right]}_{\triangleq \psi_m\left(\tau - n2^m\right)}d\tau,$$

$$= \int_{-\infty}^{\infty} x(\tau)\,\psi_m\left(\tau - n2^m\right)d\tau,$$

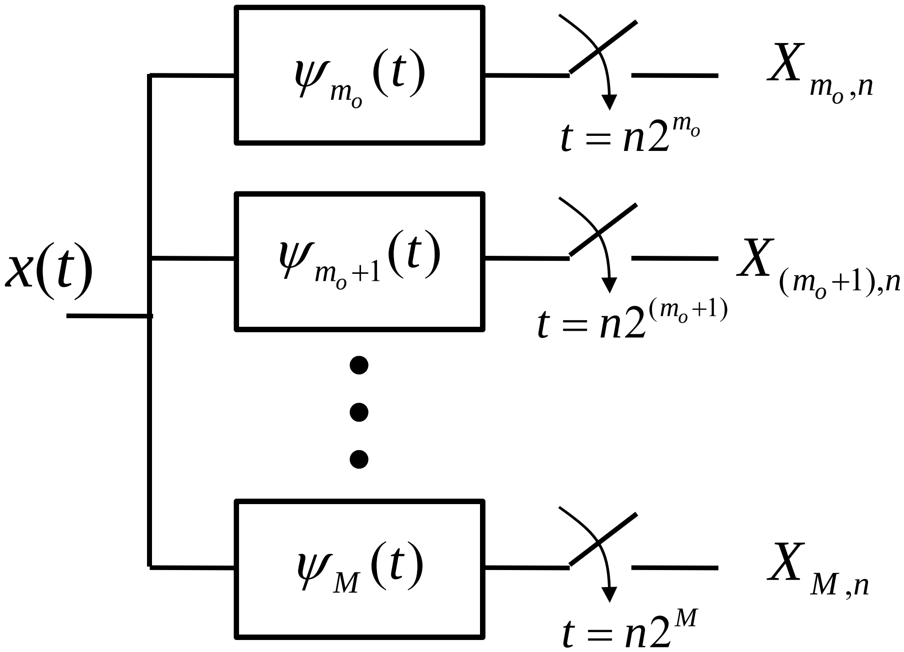
$$= \left(x * \psi_m\right)\left(n2^m\right)$$

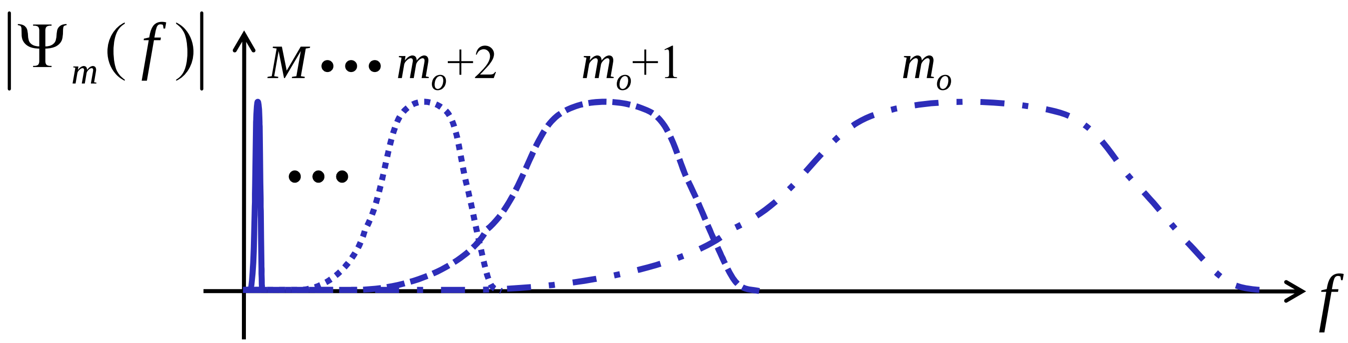


For fixed m, $X_{m,n}$ is $x(t)$ convolved with $\psi_m(t)$ and sampled at points $n2^m$.

Remember: $\psi_m(t)$ is a bandpass signal… so this is equivalent to filtering $x(t)$ with a BPF.

This leads to a filterbank:

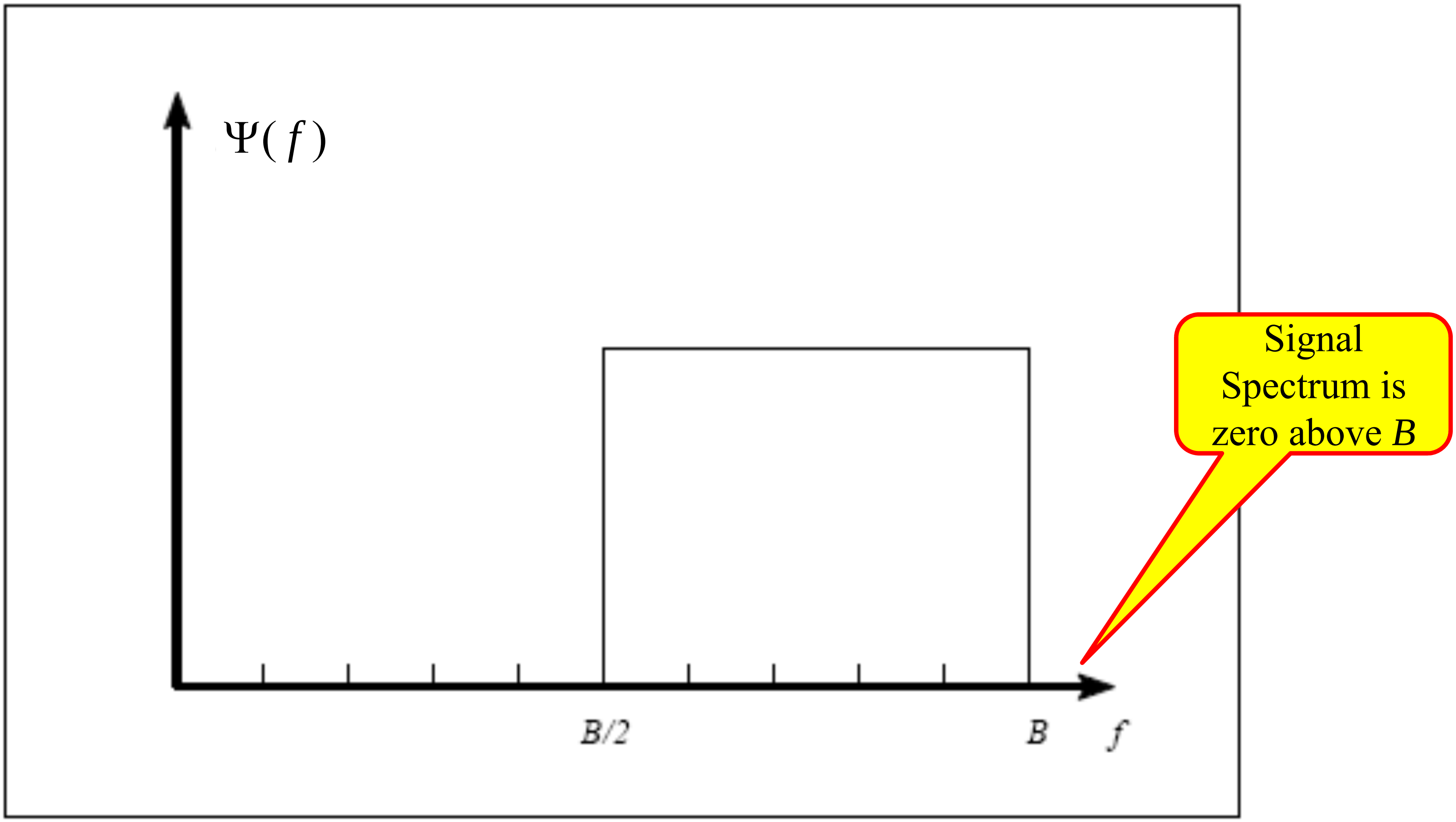




It ***IS*** possible to implement this filterbank in DT…
…but in general it is not possible to *efficiently* implement it

Consider a Special Case, Though…

Let $x(t)$ be bandlimited to B Hz. Choose the mother wavelet to be a modulated sinc function and m_o such that the spectrum is as below



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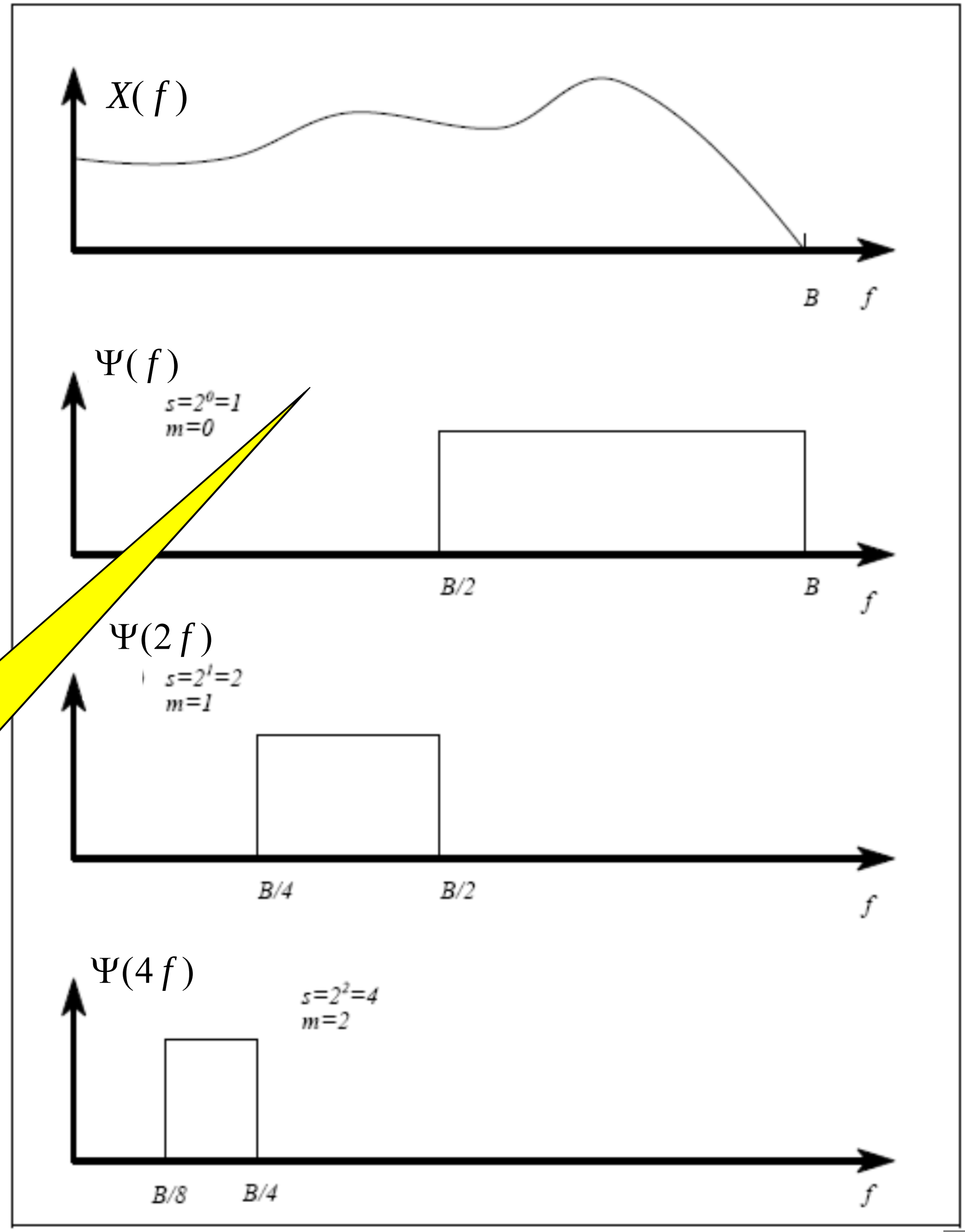
Now consider the relationship between the filter spectrum and spectra of the scaled versions of the mother wavelet…

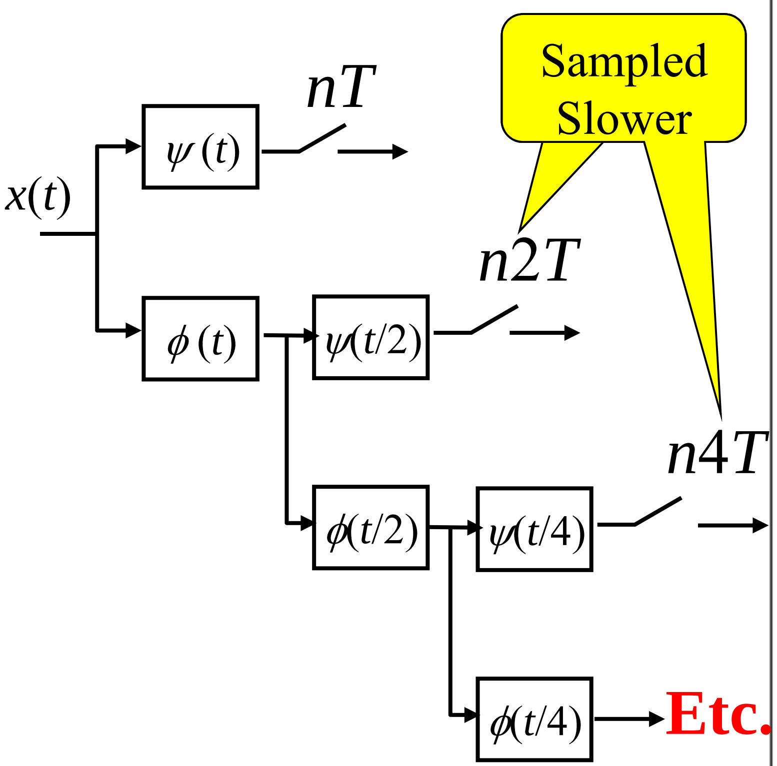


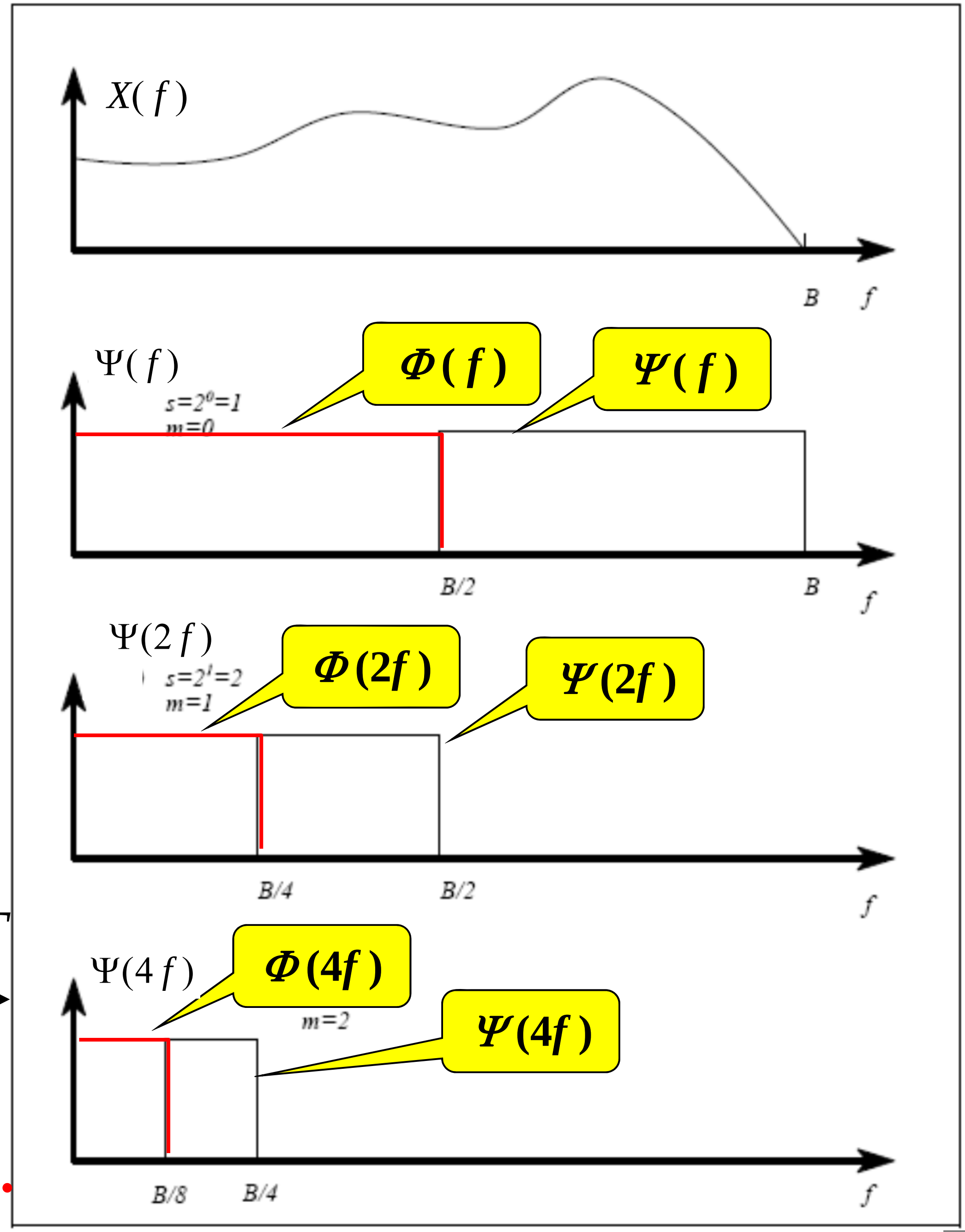
Don’t have to go to finer scale (i.e., smaller *s*… more negative *m*) because the BL signal has no power there!

Finest scale we need is on the order of $1/F_S$

Note that for these <u>ideal filters</u> we can do this…

Running the signal through $\Psi(2f)$ is the same as running it first through $\Phi(f)$ and ***then*** through $\Psi(2f)$… etc.





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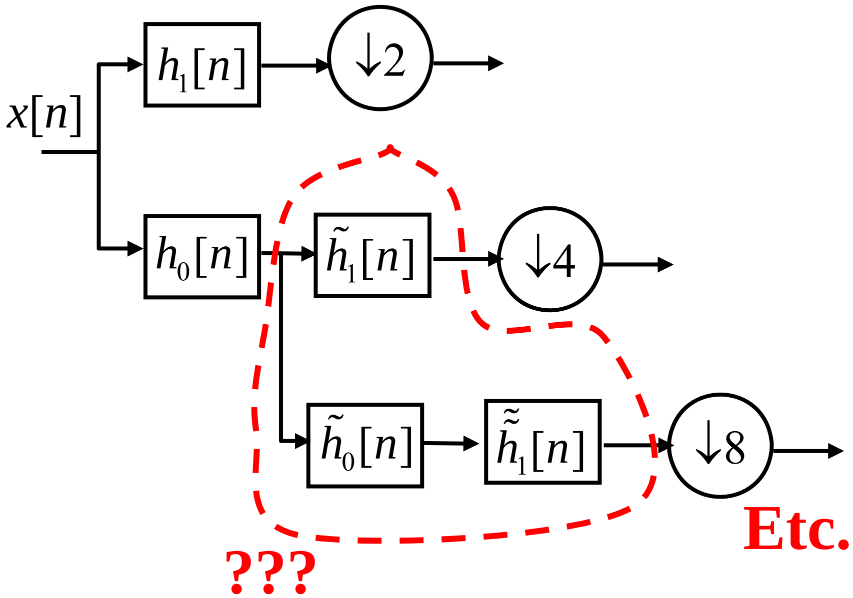
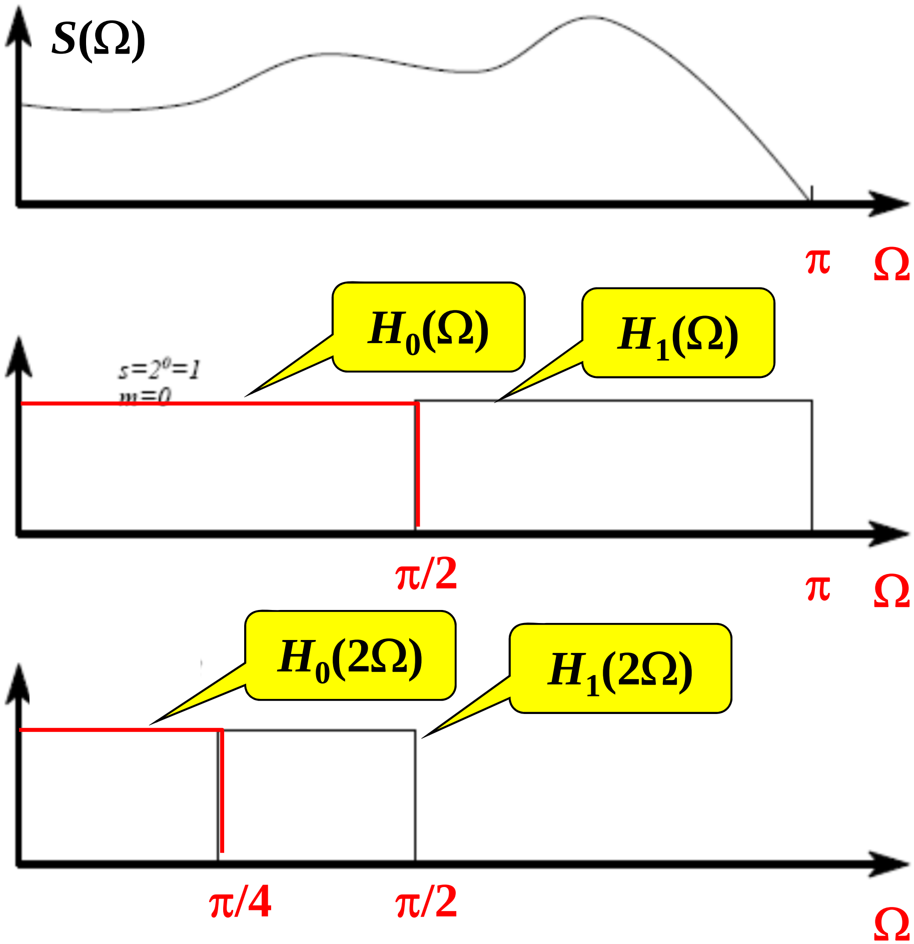
Note that we can do this in terms of DT processing…



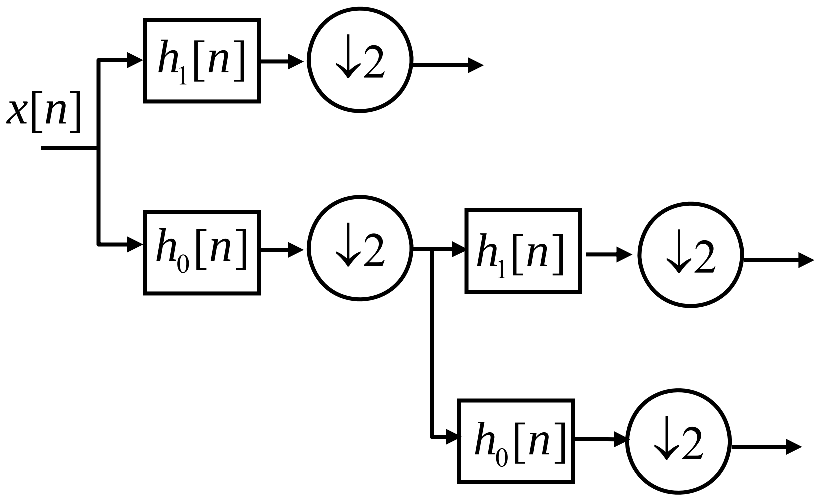


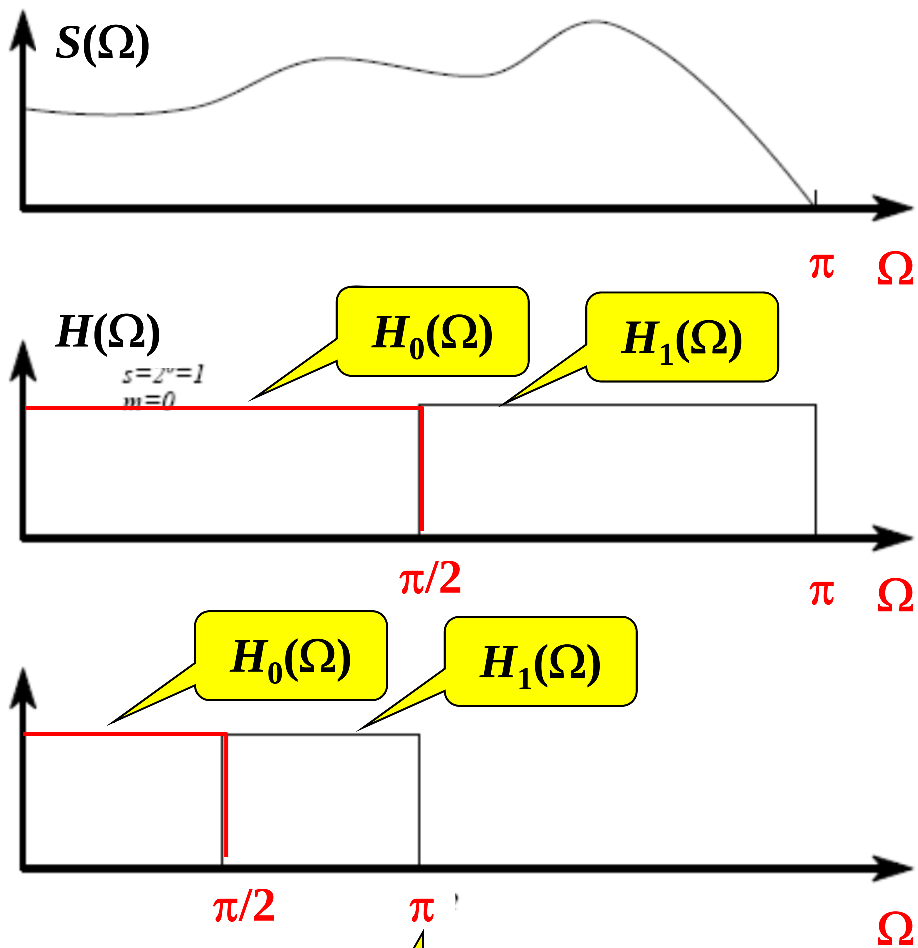
The "tildes" are used to indicate filter s that correspond to a time-scaled version of the filter w/o the tilde

$h_1[n]$ and $h_0[n]$ implement the DT versions of $\psi(t)$ and $\phi(t)$, respectively. The change in notation to "h" is to allow for the general non-sinc case where the DT filters used are not simply DT versions of the CT filters (like they are here for the sinc case).

Next

Modification of DT approach:



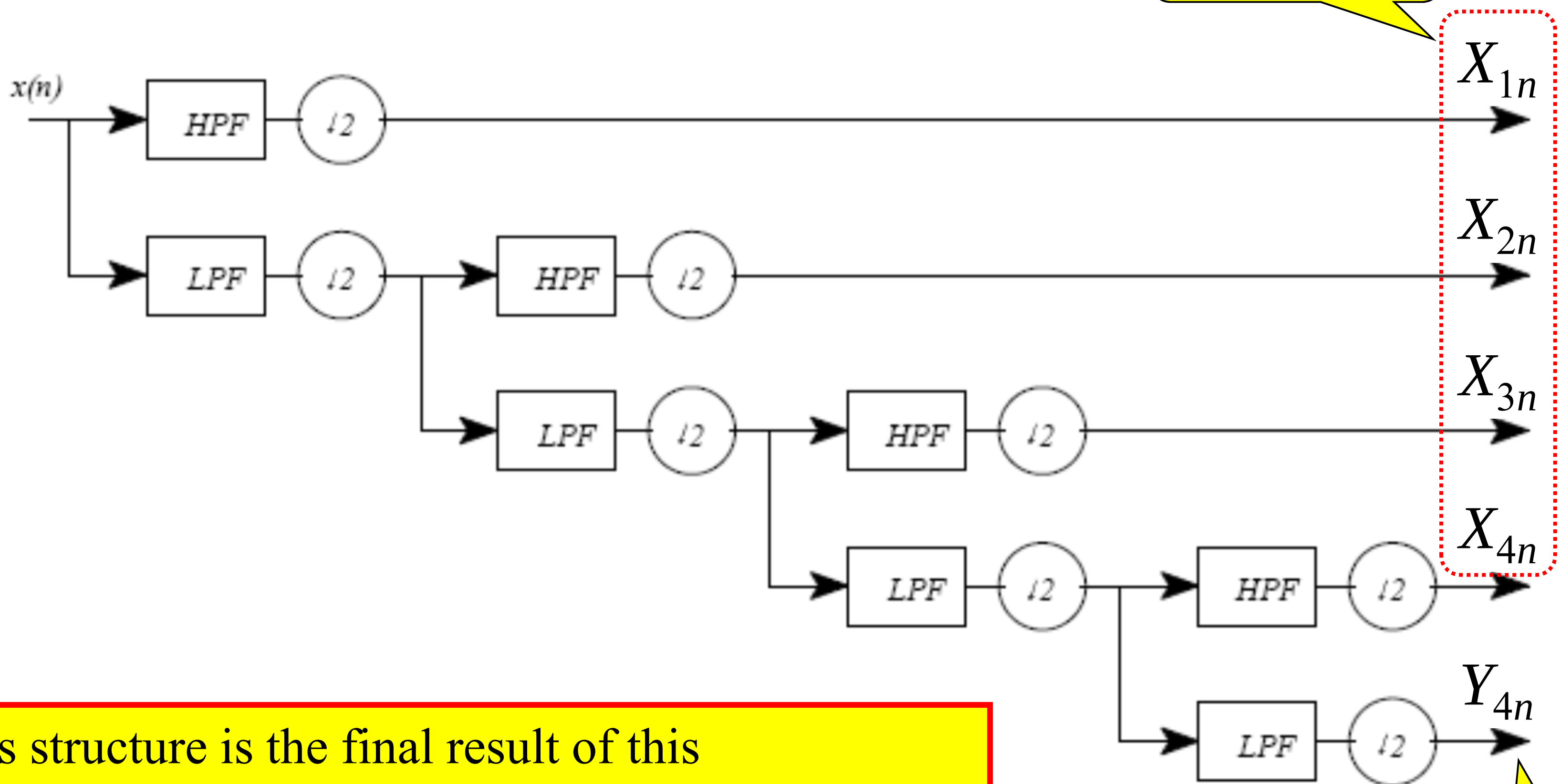


Etc.

This development is valid for the "sinc wavelet"… But the general development is not as straight-forward as this example might indicate!!!

Cascaded Filter Structure for Computing WT Coefficients

WT Coefficients



What are these??

This structure is the final result of this development… although the *way* we did the development is easily applicable only to the sinc wavelet case the result holds true but is better developed using the so-called "multi-resolution" viewpoint which we will develop next.

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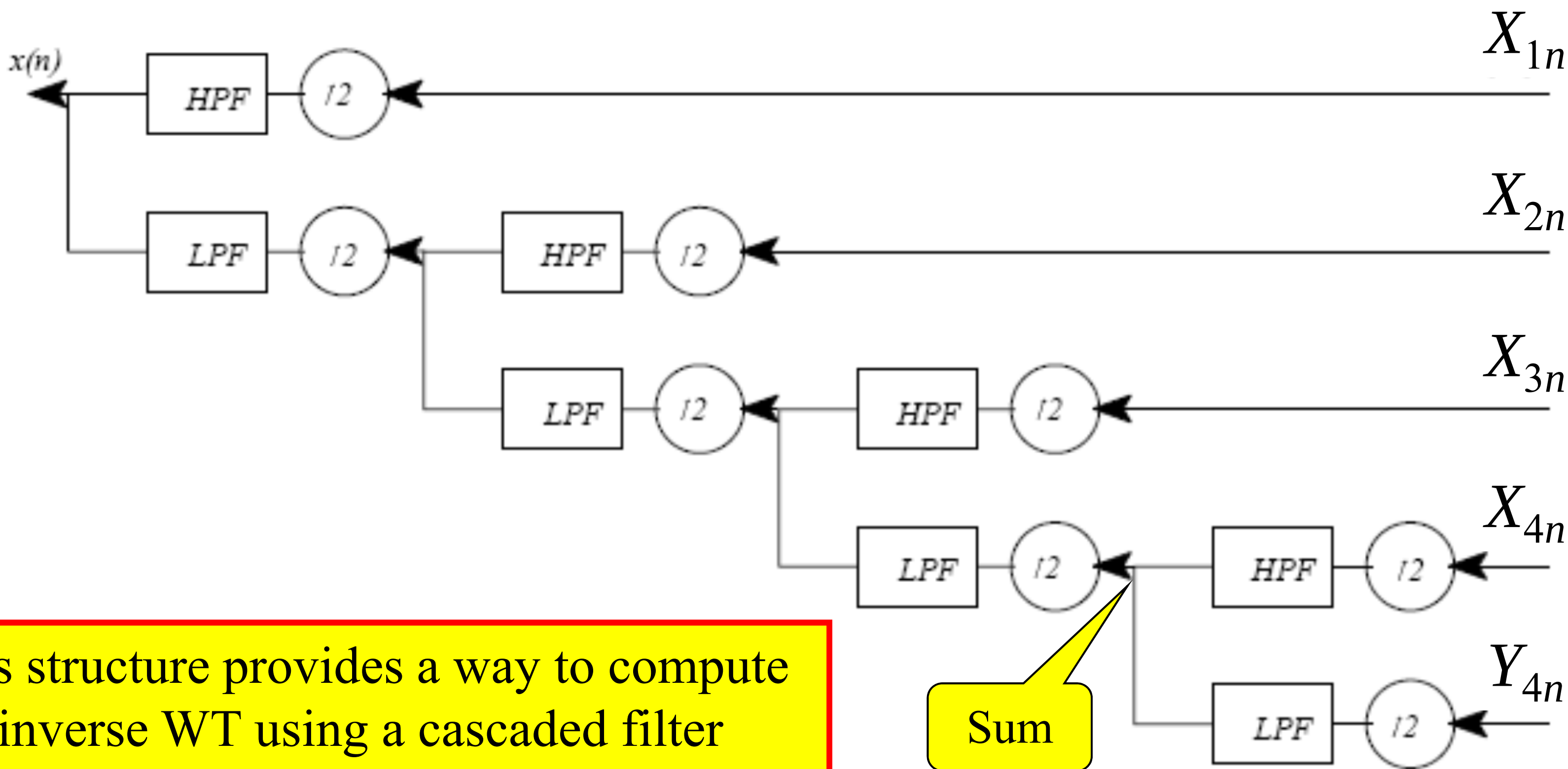
Reverse Cascaded Filter Structure for Inverting the DWT



This structure provides a way to compute the inverse WT using a cascaded filter bank structure that is essentially a reverse version of the filter bank used to compute the WT.

Next

Multi-Resolution Viewpoint

Provides solid development of DT Filter Bank Implementation

Multi-Resolution Approach

- Stems from image processing field
  - consider finer and finer approximations to an image
- Define a nested set of signal spaces

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2$$

- We build these spaces as follows:
- Let V_0 be the space spanned by the integer translations of a fundamental signal $\phi(t)$, called the ***scaling function***: spanned by $\phi(t-k)$

- So far we can use just about any function $\phi(t)$, but we'll see that to get the nesting only certain scaling functions can be used.

Next

Multiresolution Analysis (MRA) Equation

- Now that we have V_0 how do we make the others and ensure that they are nested?
- If we let V_1 be the space spanned by integer translates of $\phi(2t)$ we get the desired property that V_1 is indeed a space of functions having higher resolution.
- Now how do we get the nesting?
- We need that any function in V_0 also be in V_1; in particular we need that the scaling function (which is in V_0) be in V_1, which then requires that

$$\phi(t) = \sum_n h_0[n]\sqrt{2}\phi(2t-n)$$

  where the expansion coefficient is $h_0[n]2^{½}$

- This is the requirement on the scaling function to ensure nesting: it must satisfy this equation
  - called the multiresolution analysis (MRA) equation
  - this is like a differential equation for which the scaling function is the solution

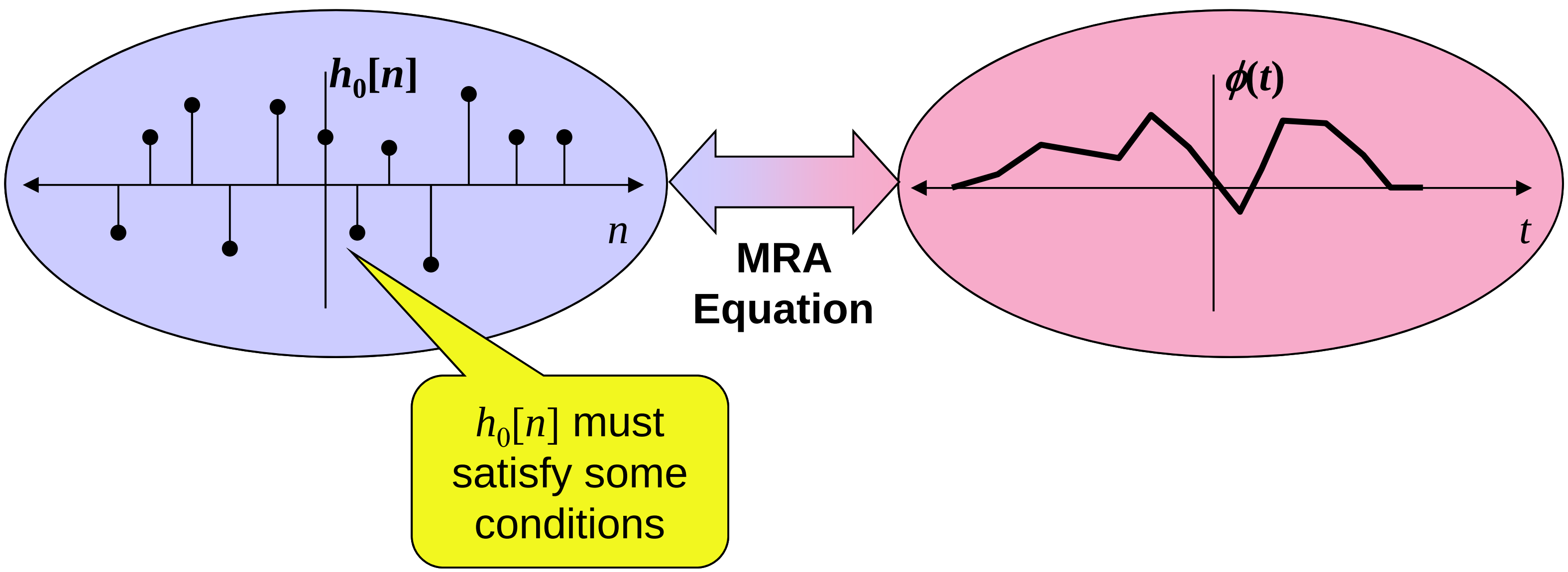
Next

The $h_0[n]$ Specify the Scaling Function

- Thus, the coefficients $h_0[n]$ determine the scaling function
  - for a given set of $h_0[n]$, $\phi(t)$
    - may or may not exist
    - may or may not be unique

$$\phi(t) = \sum_n h_0[n]\sqrt{2}\phi(2t-n)$$

- Want to find conditions on $h_0[n]$ for $\phi(t)$ to exist and be unique, and also:
  - to be **orthogonal** (because that leads to an ON wavelet expansion)
  - to give wavelets that have **desirable properties**



Whence the Wavelets?

- The spaces V_j represent increasingly higher resolution spaces
- To go from V_j to higher resolution V_{j+1} requires the addition of "details"
  - These details are the part of V_{j+1} not able to be represented in V_j
  - This can be captured through the "orthogonal complement" of V_j w.r.t V_{j+1}
- Call this orthogonal complement space W_j
  - all functions in W_j are orthogonal to all functions in V_j
  - That is:

$$<\phi_{j,k}(t),\psi_{j,l}(t)>=\int \phi_{j,k}(t)\psi_{j,l}(t)dt = 0 \qquad \forall j,k,l \in \mathbf{Z}$$

- Consider that V_0 is the lowest resolution of interest
- How do we characterize the space W_0 ?
  - we need to find an ON basis for W_0, say $\{\psi_{0,k}(t)\}$ where the basis functions arise from translating a single function (we'll worry about the scaling part later):

$$\psi_{0,k}(t) = \psi(t-k)$$

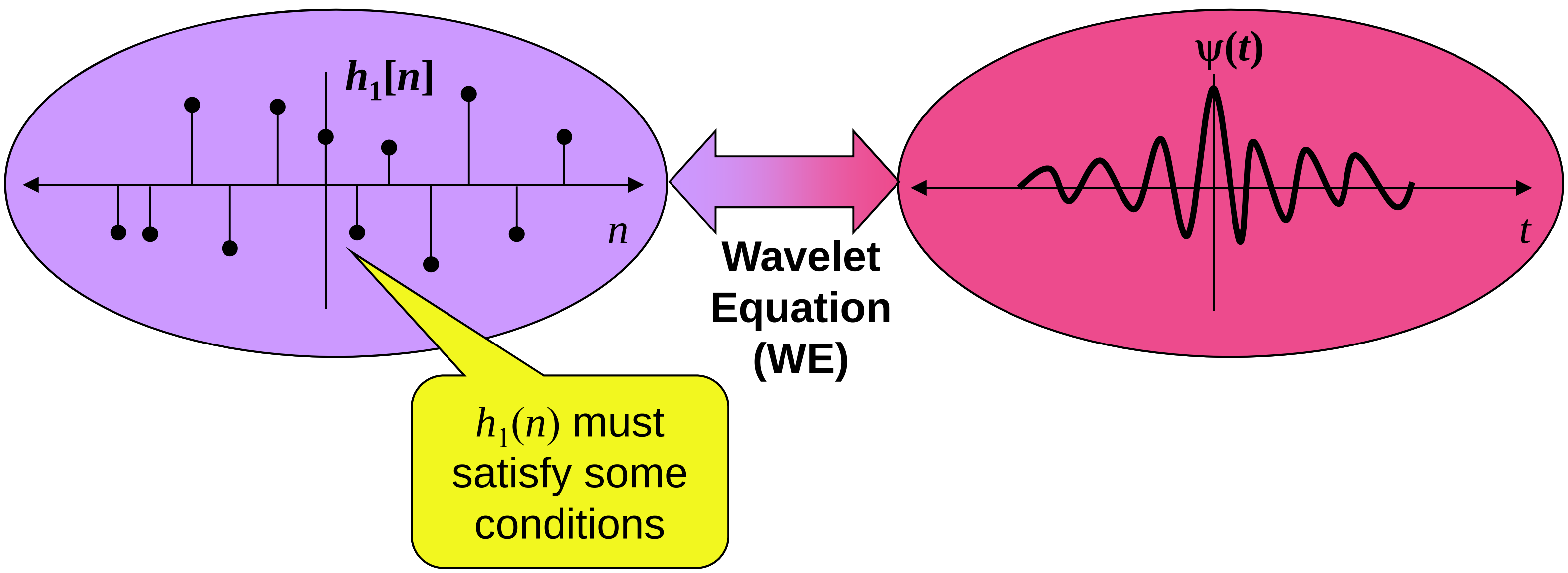
Next

Finding the Wavelets

- The wavelets are the basis functions for the W_j spaces
  - thus, they lie in V_{j+1}
- In particular, the function $\psi(t)$ lies in the space V_1 so it can be expanded as

$$\psi(t) = \sum_n h_1[n]\sqrt{2}\phi(2t-n), \quad n \in \mathbf{Z}$$

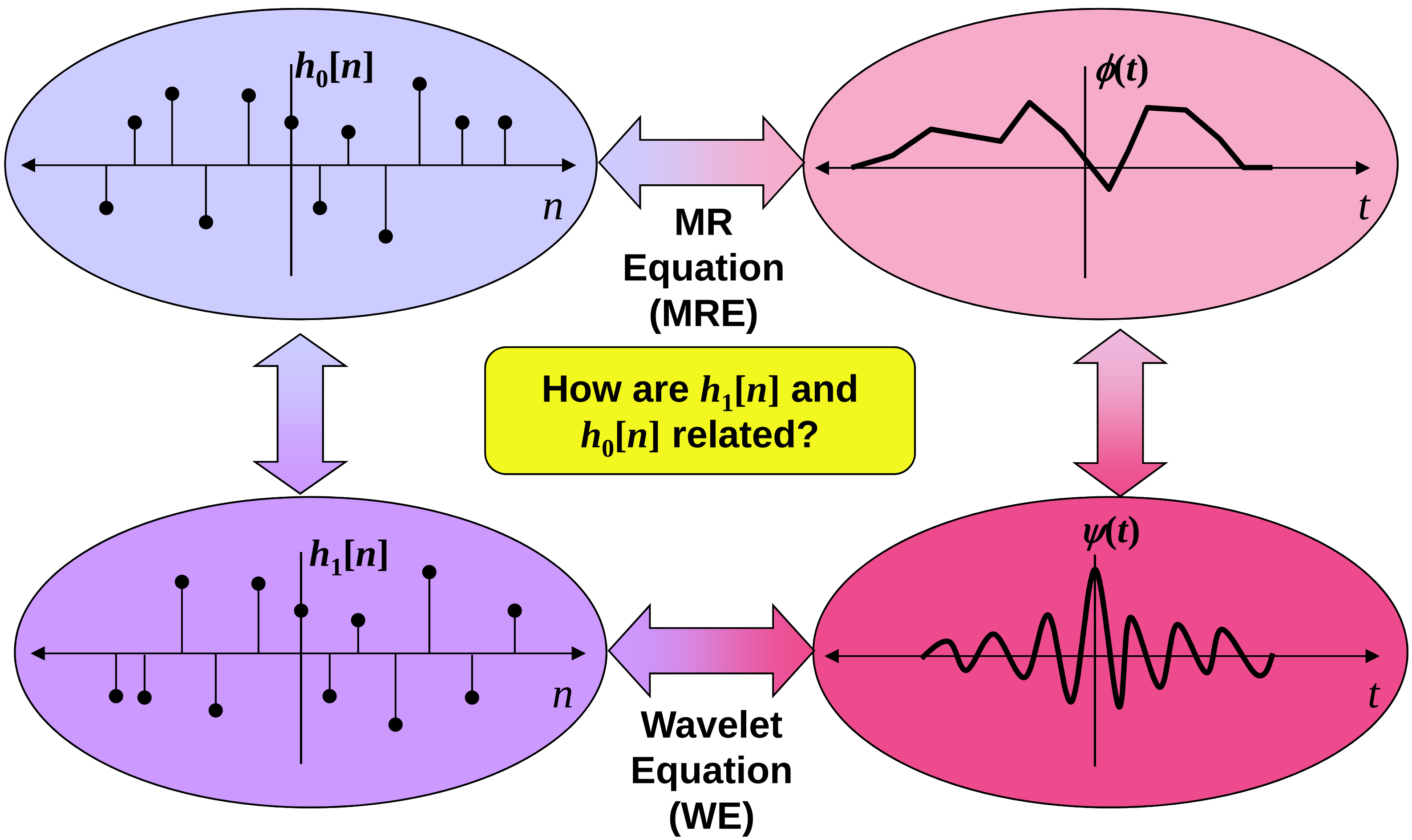
- This is a fundamental result linking the scaling function and the wavelet
  - the $h_1[n]$ specify the wavelet, via the specified scaling function



Next

Wavelet-Scaling Function Connection

- There is a fundamental connection between the scaling function and its coefficients $h_0[n]$, the wavelet function and its coefficients $h_1[n]$:



Next

Relationship Between $h_1[n]$ and $h_0[n]$

- We state here the conditions for the important special case of
  - finite number N+1 of nonzero $h_0[n]$
  - ON within V_0: $\int \phi(t)\phi(t-k)dt = \delta(k)$
  - ON between V_0 and W_0 : $\int \psi(t)\phi(t-k)dt = \delta(k)$
- Given the $h_0[n]$ that define the desired scaling function, then the $h_1[n]$ that define the wavelet function are given by

$$h_1[n] = (-1)^n h_0[N-n]$$

where N is the "order" of the "filter"

We'll see soon that the "h" coefficients are really DT filter coefficients

- Much of wavelet theory addresses the origin, characteristics, and ramifications of this relationship between $h_1[n]$ and $h_0[n]$
  - requirements on $h_0[n]$ and $h_1[n]$ to achieve ON expansions
  - how the MRE and WE lead to a filter bank structure
  - requirements on $h_0[n]$ and $h_1[n]$ to achieve other desired properties
  - extensions beyond the ON case

The Resulting Expansions

- Suppose we have found a scaling function $\phi(t)$ that satisfies the MRE
- Then… $\phi(t - k)$ is an ON basis for V_0
- More generally, an ON basis for V_{j_o} is $\left\{2^{j_o/2}\phi(2^{j_o}t-k)\right\}_{k=-\infty}^{\infty}$

- Since V_{jo} is a subspace of $L^2(R)$ we can find the "best approximation" to $x(t)\in L^2(R)$ as follows

$$\boxed{x_{j_o}(t)=\sum_k c_{j_o,k}2^{j_o/2}\varphi(2^{j_o}t-k)}$$

Not the most useful expansion in practice

with $$c_{j_o,k}=\left\langle x(t),\phi_{j_o,k}(t)\right\rangle=\int_{-\infty}^{\infty}x(t)2^{j_o/2}\phi(2^{j_o}t-k)dt$$

$x_{jo}(t)$ is a low-resolution approximation to $x(t)$

Increasing j_o gives a better (i.e., higher resolution) approximation

$$\cdots\subset V_{-2}\subset V_{-1}\subset V_0\subset V_1\subset V_2\subset\cdots\subset L^2$$

Next

The Resulting Expansions (cont.)

- We've set things up so that for some j_o and its space V_{jo} we have that

$$L^2 = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus W_{j_0+2} \oplus \cdots$$

- We know that an ON basis for V_{j_o} is $\left\{2^{j_o/2}\phi(2^{j_o}t-k)\right\}_{k=-\infty}^{\infty}$
- We also know than on ON basis for W_j is $\left\{2^{j/2}\psi(2^{j}t-k)\right\}_{k=-\infty}^{\infty}$
- Thus we another form of the expansion:

$$x(t) = \underbrace{\sum_k c_{j_0,k} 2^{j_0/2}\varphi(2^{j_0}t-k)}_{\text{Low-Resolution Approximation}} + \underbrace{\sum_k \sum_{j=j_0}^{\infty} d_{j,k} 2^{j/2}\psi(2^{j}t-k)}_{\text{Wavelet Details}}$$

A ***VERY*** useful expansion in practice

$$c_{j_o,k} = \left\langle x(t), \phi_{j_o,k}(t) \right\rangle = \int_{-\infty}^{\infty} x(t) 2^{j_o/2}\phi(2^{j_o}t-k)dt$$

$$d_{j,k} = \left\langle x(t), \psi_{j,k}(t) \right\rangle = \int_{-\infty}^{\infty} x(t) 2^{j/2}\psi(2^{j}t-k)dt$$

Same as the $X_{j,k}$ WT coefficients in the earlier notes

Next

The Resulting Expansions (cont.)

- If we let j_o go to $-\infty$ then

$$L^2 = \cdots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots$$

- And… the above expansion becomes

$$x(t) = \sum_k \sum_{j=-\infty}^{\infty} d_{j,k} 2^{j/2} \psi(2^j t - k)$$

Not the most useful expansion in practice

$$d_{j,k} = \left\langle x(t), \psi_{j,k}(t) \right\rangle = \int_{-\infty}^{\infty} x(t) 2^{j/2} \psi(2^{j_o} t - k) dt$$

- This is most similar to the "true" wavelet decomposition as it was originally developed

Next

The Expansion Coefficients $c_{j_0}(k)$ and $d_j(k)$

- We consider here only the simple, but important, case of ON expansion
  - i.e., the ϕ's are ON, the ψ's are ON, <u>*and*</u> the ϕ's are ON to the ψ's
- Then we can use standard ON expansion theory:

$$c_{j_0,k} = \left\langle x(t), \phi_{j_0,k}(t) \right\rangle = \int x(t)\phi_{j_0,k}(t)\,dt$$

$$d_{j,k} = \left\langle x(t), \psi_{j,k}(t) \right\rangle = \int x(t)\psi_{j,k}(t)\,dt$$

- We will see how to compute these without resorting to computing inner products
  - we will use the coefficients $h_1[n]$ and $h_0[n]$ instead of the wavelet and scaling function, respectively
  - we look at a relationship between the expansion coefficients at one level and those at the next level of resolution

Next

Filter Banks and DWT

Next

Generalizing the MRE and WE

- Here again are the MRE and the WE:

$$\phi(t) = \sum_n h_0[n]\sqrt{2}\phi(2t-n) \qquad \psi(t) = \sum_n h_1[n]\sqrt{2}\phi(2t-n)$$

scale & translate: replace $t \rightarrow 2^j t - k$

- We get:

MRE

$$\phi(2^j t - k) = \sum_m h_0[m-2k]\sqrt{2}\phi(2^{j+1}t - m)$$

Connects V_j to V_{j+1}

WE

$$\psi(2^j t - k) = \sum_m h_1[m-2k]\sqrt{2}\phi(2^{j+1}t - m)$$

Connects W_j to V_{j+1}

Linking Expansion Coefficients Between Scales

- Start with the Generalized MRA and WE:

$$\phi(2^j t - k) = \sum_m h_0[m-2k]\sqrt{2}\phi(2^{j+1}t - m) \qquad \psi(2^j t - k) = \sum_m h_1[m-2k]\sqrt{2}\phi(2^{j+1}t - m)$$

$$c_{j,k} = \langle x(t), \phi_{j,k}(t) \rangle \qquad d_{j,k} = \langle x(t), \psi_{j,k}(t) \rangle$$

$$c_{j,k} = \sum_m h_0[m-2k] \langle x(t), 2^{(j+1)/2}\phi(2^{j+1}t - m) \rangle \qquad d_{j,k} = \sum_m h_1[m-2k] \langle x(t), 2^{(j+1)/2}\phi(2^{j+1}t - m) \rangle$$

$$c_{j+1}(m)$$

$$c_{j,k} = \sum_m h_0[m-2k]\, c_{j+1,m} \qquad d_{j,k} = \sum_m h_1[m-2k]\, c_{j+1,m}$$

Convolution-Decimation Structure

$$c_{j,k} = \sum_m h_0[m-2k]\, c_{j+1,m}$$

$$d_{j,k} = \sum_m h_1[m-2k]\, c_{j+1,m}$$

Convolution

$$y_0[n] = c_{j+1}[n] * h_0[-n] = \sum_m h_0[m-n]\, c_{j+1}[m]$$

$$y_1[n] = c_{j+1}[n] * h_1[-n] = \sum_m h_1[m-n]\, c_{j+1}[m]$$

Decimation

● ● ● ● ● ● ● ● ● ●

$n = 0$ 1 2 3 4 5 6 7 8 9

● ○ ● ○ ● ○ ● ○ ● ○

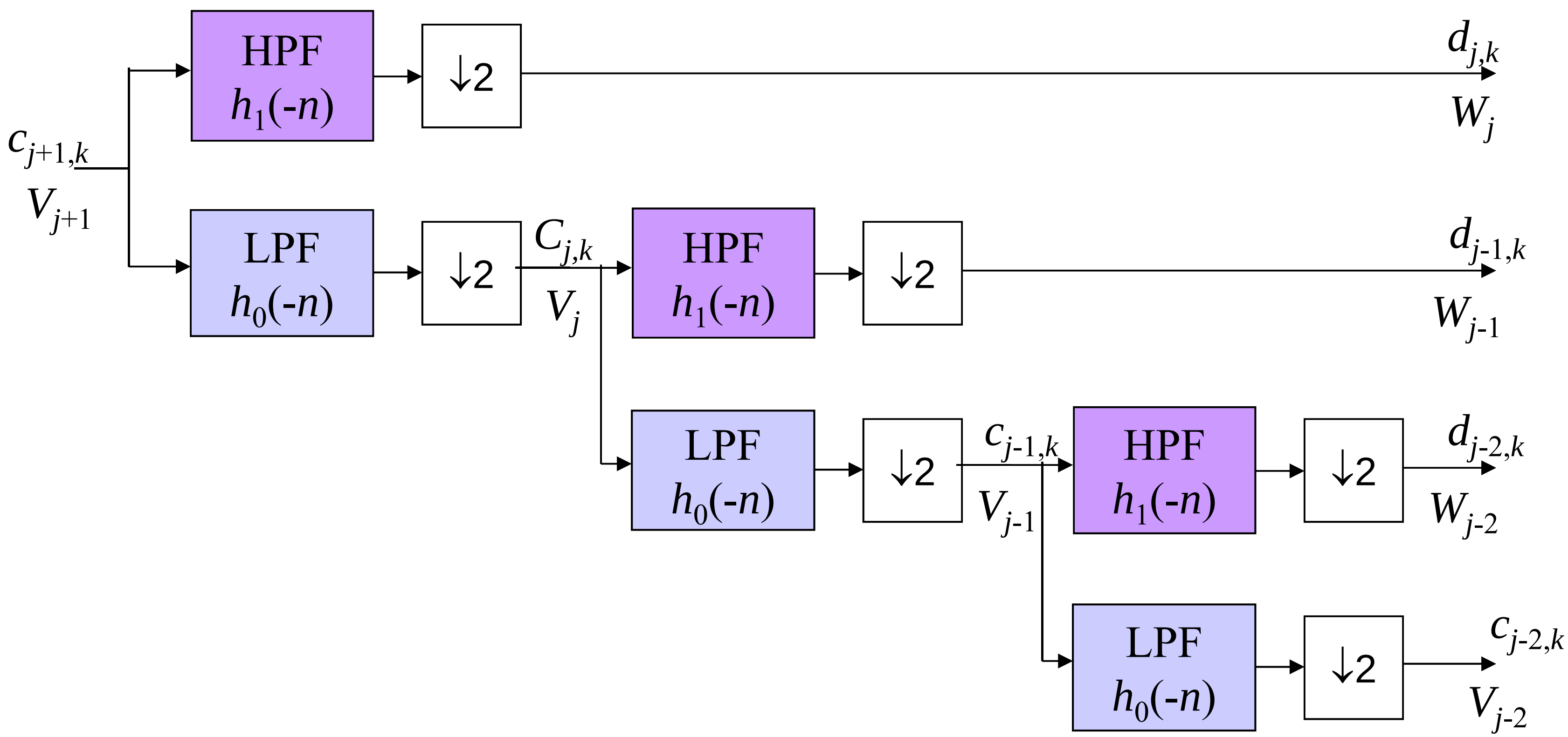
$k = 0$ 1 2 3 4

$n = 2k = 0$ 2 4 6 8

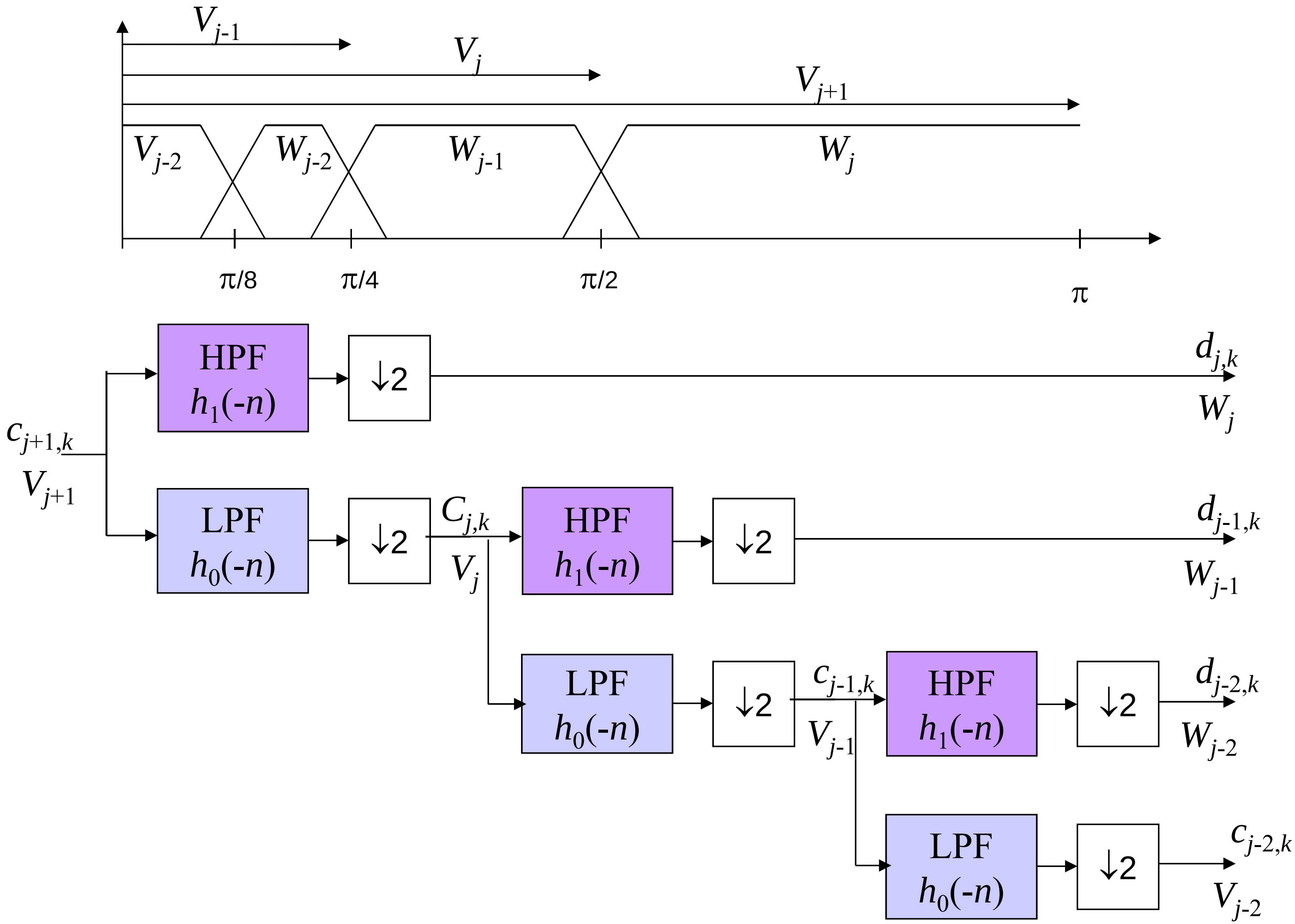
Next

Computing The Expansion Coefficients

- The above structure can be cascaded:
  - given the scaling function coefficients at a specified level all the lower resolution c's and d's can be computed using the filter structure

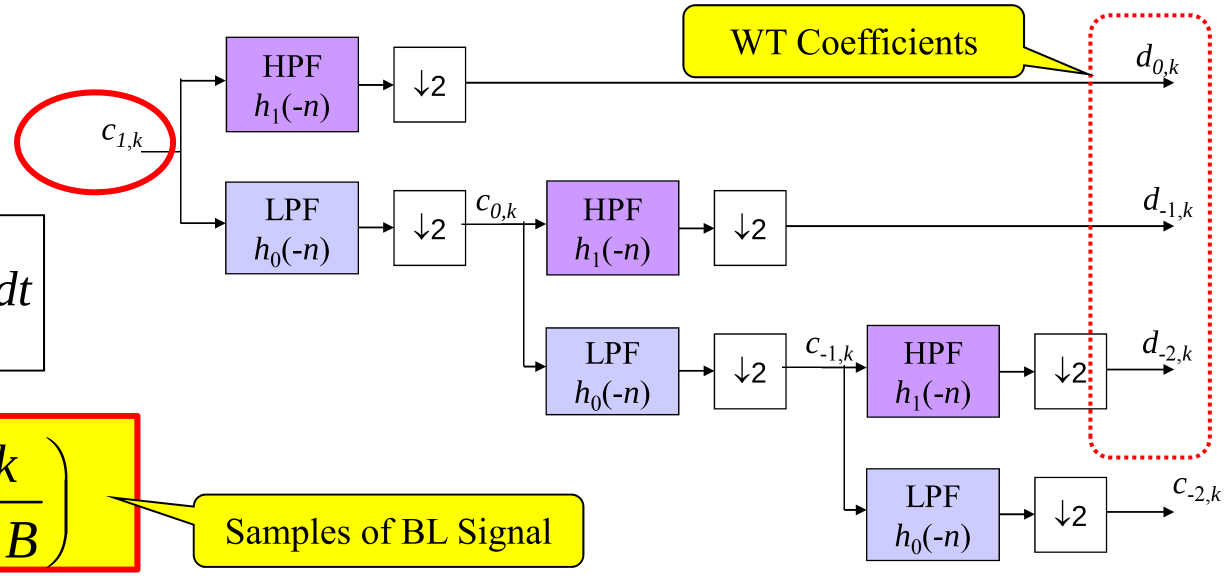


Filter Bank Generation of the Spaces



Next

How do we get these???!!!

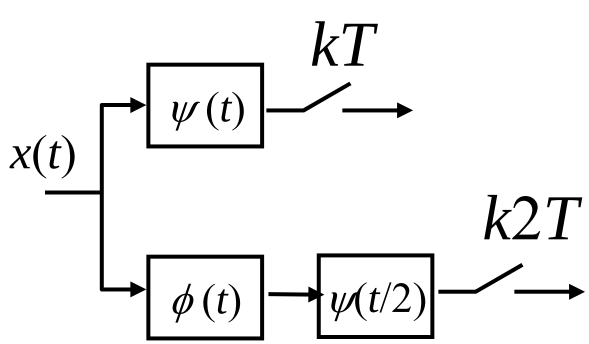


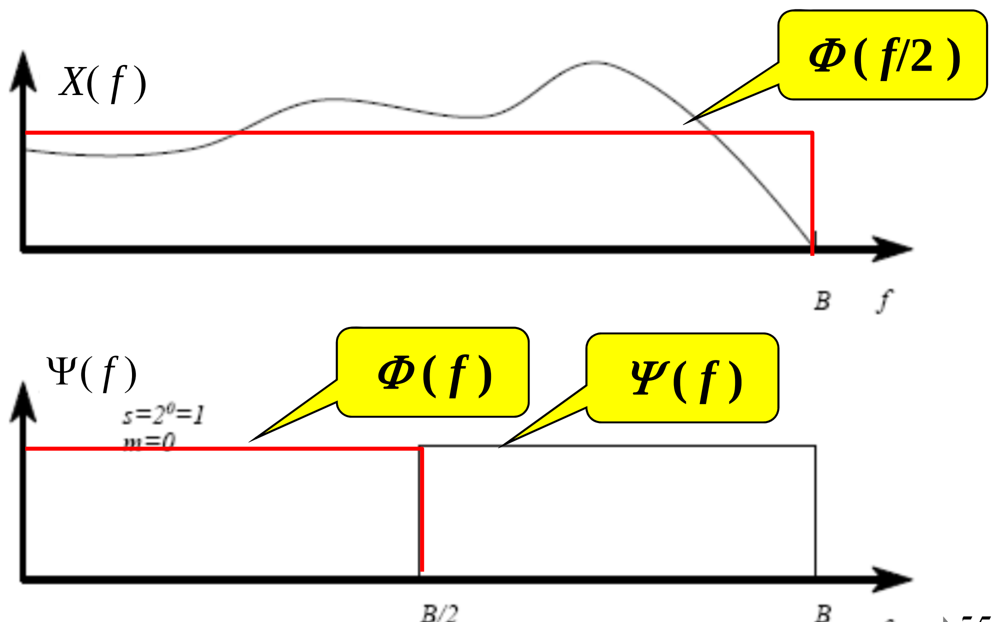
$$c_{1,k} = \int_{-\infty}^{\infty} x(t) 2^{1/2} \phi(2t-k)\,dt$$

Conjecture: $c_{j+1,k} \approx x\left(\frac{k}{2B}\right)$

Samples of BL Signal

Consider case where $\phi(2t-k) = \mathrm{sinc}(2Bt-k) = \mathrm{sinc}\left(2B(t-k/2B)\right)$





Next

$$\phi(2t-k)=\mathrm{sinc}(2Bt-k)$$

First, prove this: $$x\left(\frac{n}{2B}\right)=\int_{-\infty}^{\infty}x(t)\,\mathrm{sinc}(2Bt-k)\,dt$$

$$\int_{-\infty}^{\infty}x(t)\,\mathrm{sinc}(2Bt-k)\,dt=\int_{-\infty}^{\infty}x(t)\,\mathrm{sinc}\big(2B(t-k/2B)\big)\,dt$$

FT of sinc = rectangle

Generalized Parseval's Theorem

$$=\int_{-B}^{B}X(f)\times 1\times e^{j2\pi f\frac{k}{2B}}df$$

Pure Cleverness!!

$$=\left[\int_{-B}^{B}X(f)e^{j2\pi ft}e^{j2\pi f\frac{k}{2B}}df\right]_{t=0}$$

$$=\left[\int_{-B}^{B}X(f)e^{j2\pi f(t+\frac{k}{2B})}df\right]_{t=0}$$

Inverse FT

$$=\left[x(t+\tfrac{k}{2B})\right]_{t=0}$$

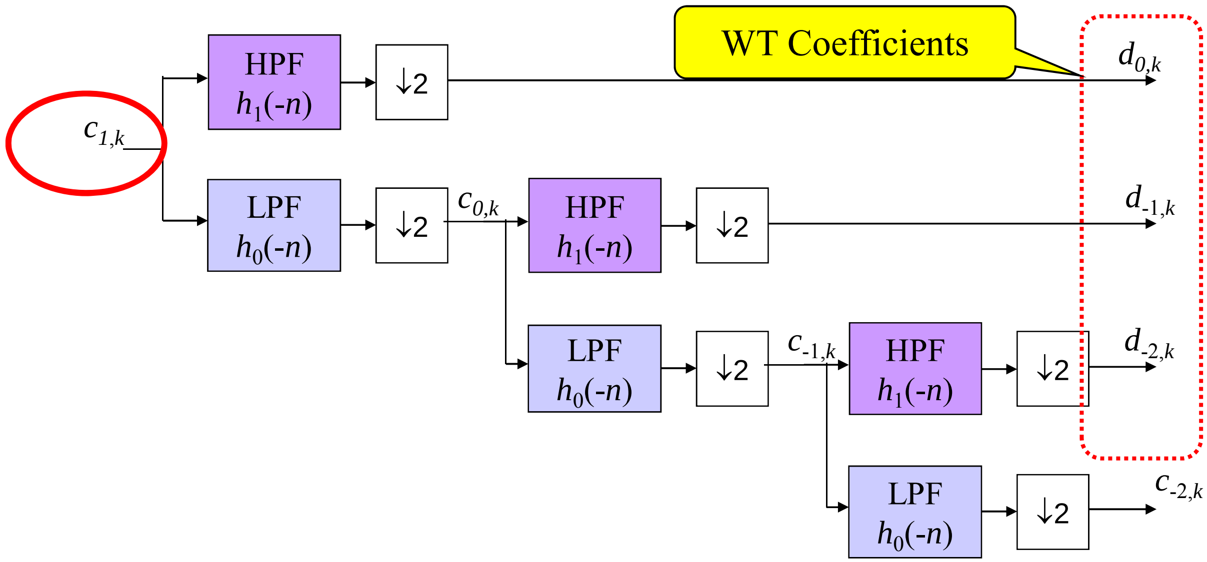
$$=x(\tfrac{k}{2B})$$

So… for the "sinc wavelet" case… the conjecture is true with perfect equality

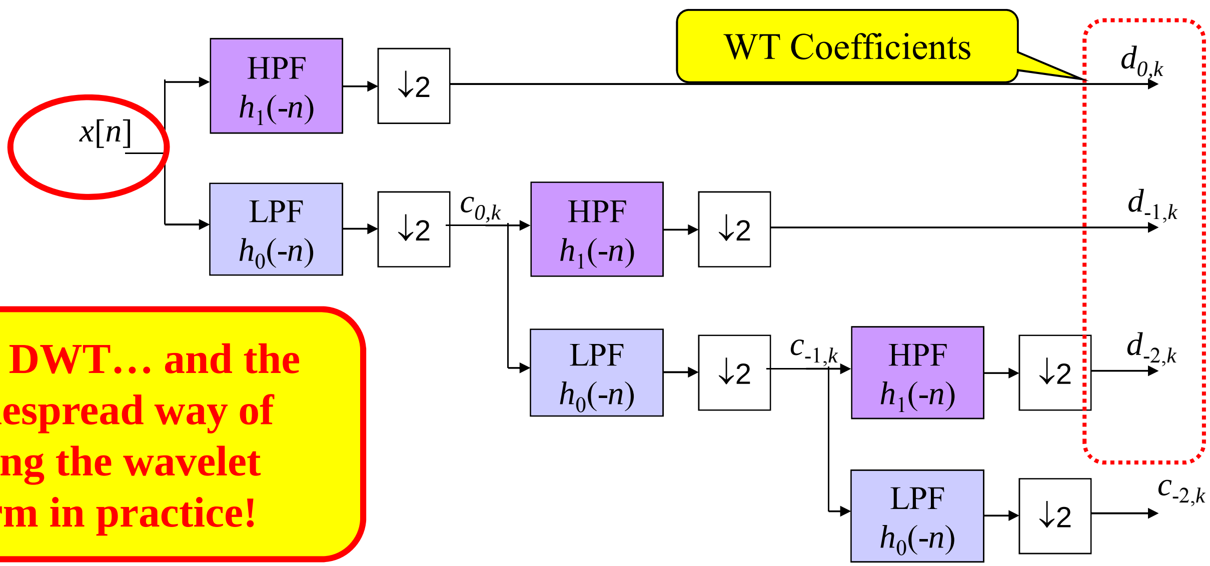
For other cases… there is some high enough scale where this result holds approximately!

Next

So… now we can change this…



Into this…



This is the DWT… and the most widespread way of computing the wavelet transform in practice!

Next

Connection between Notations of WT Notes

$$X(s,\tau)=\int_{-\infty}^{\infty} x(t)\left[\frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)\right]dt$$

$$x(t)=\int_{0}^{\infty}\int_{-\infty}^{\infty} X(s,\tau)\left[\frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)\right]\frac{ds\,d\tau}{s^2}$$

$$X_{mn}=\int_{-\infty}^{\infty} x(t)\left[2^{-m/2}\psi\left(2^{-m}t-n\right)\right]dt$$

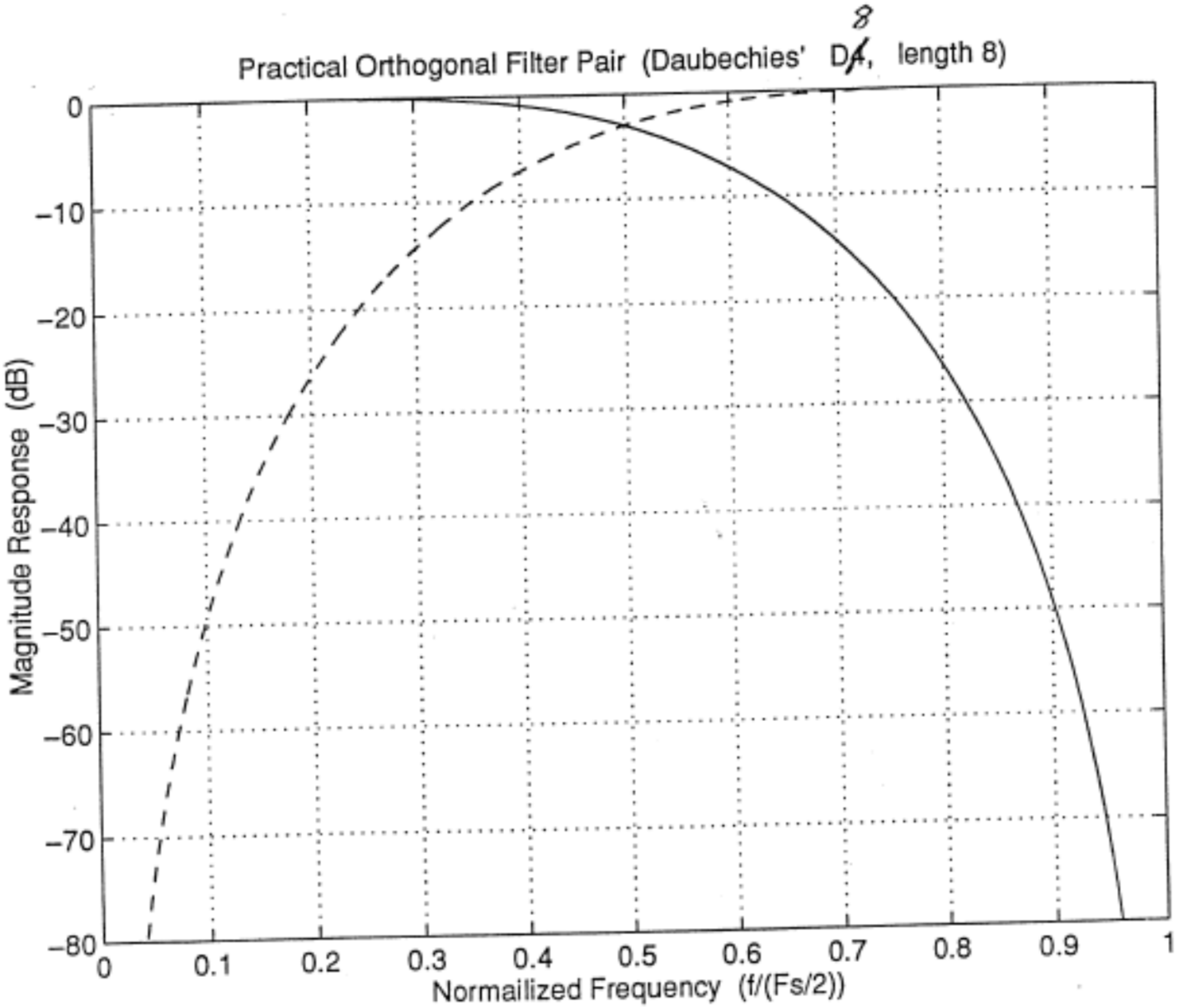
$$x(t)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} X_{mn}\left[2^{-m/2}\psi\left(2^{-m}t-n\right)\right]$$

Compute $d_j(k)$… & $c_j(k)$…

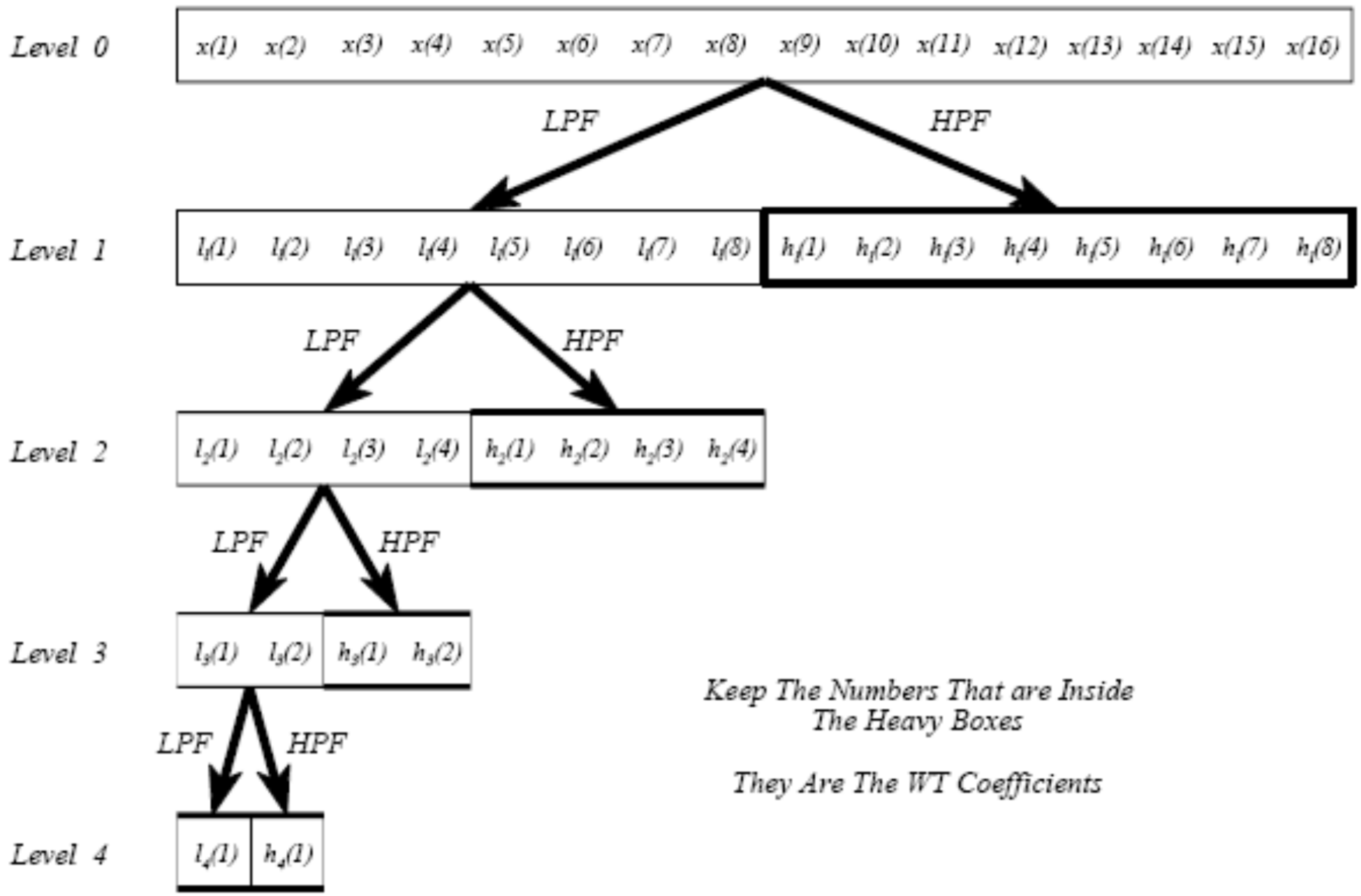
using filter bank

$$x(t)=\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} d_j(k)\left[2^{j/2}\psi\left(2^{j}t-k\right)\right]$$

$$x(t)=\sum_{k} c_{j_0,k}\,2^{j_0/2}\varphi(2^{j_0}t-k)+\sum_{k}\sum_{j=j_0}^{\infty} d_{j,k}\,2^{j/2}\psi(2^{j}t-k)$$

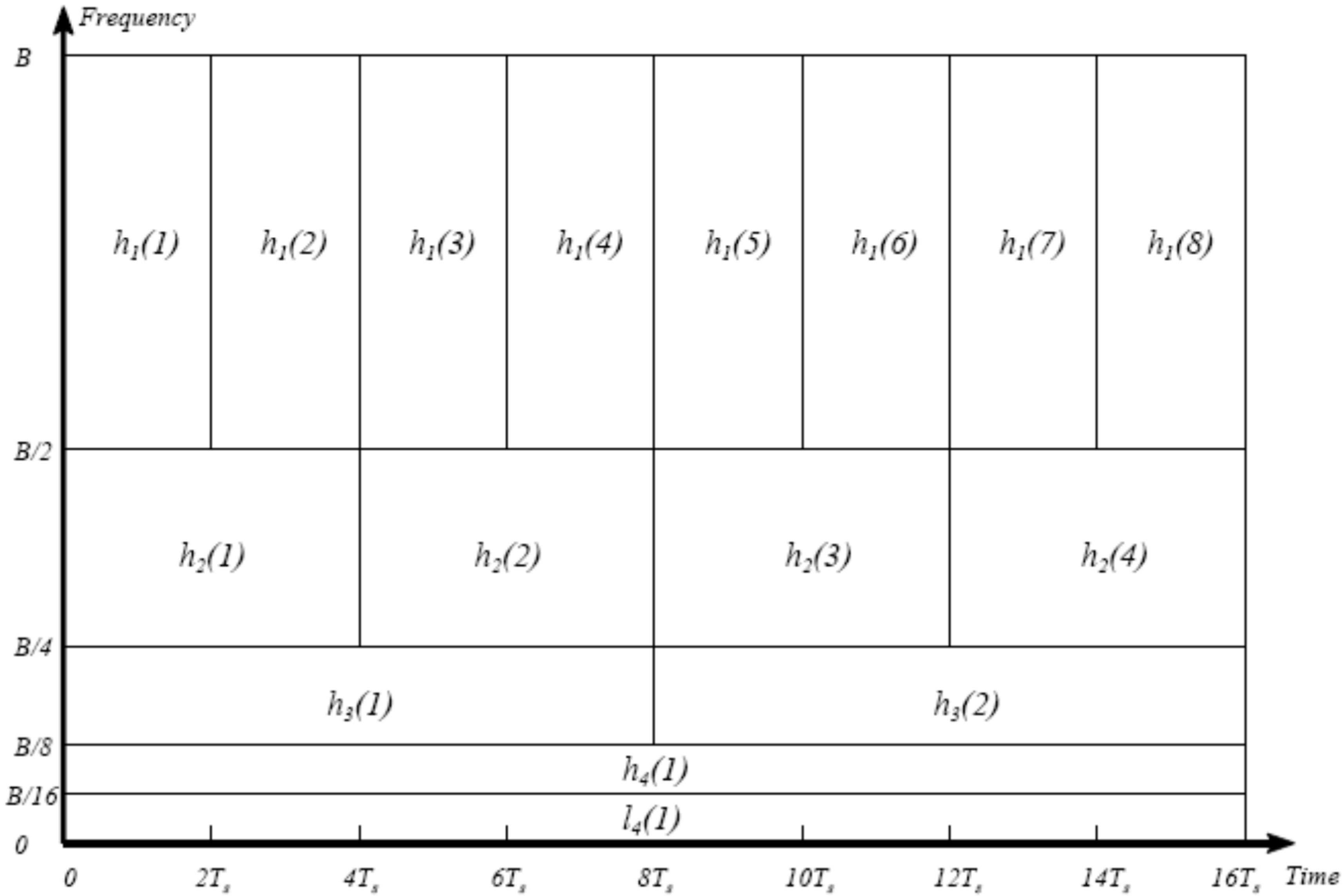

Practical Orthogonal Filter Pair (Daubechies' D4, length 8)
8
Magnitude Response (dB)
0
-10
-20
-30
-40
-50
-60
-70
-80
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normailized Frequency (f/(Fs/2))

Another Way of Viewing the WT: The Coefficient Pyramid



Next

Another Way of Viewing the WT: Time-Frequency Tiling



Next

Computational Complexity of DWT

For a signal of length *N*: # of Multiplies & Adds is *O*(*N*)

Lower order than the FFT which is $O(N\log_2 N)$

But watch out for the multiplicative constant!

Each Filter has length $L << N$

For 1st Stage:

- Each of two filters: computes $N/2$ outputs, each requiring L multiples
- # Multiplies for 1st stage = $NL/2 + NL/2 = NL$

For 2nd Stage:

- Each of two filters: computes $N/4$ outputs, each requiring L multiples
- # Multiplies for 1st stage = $NL/4 + NL/4 = NL/2$

For 3rd Stage:

- Each of two filters: computes $N/8$ outputs, each requiring L multiples
- # Multiplies for 1st stage = $NL/8 + NL/8 = NL/4$

$$\#\text{ of Mult.} < NL\sum_{m=0}^{\infty} 2^{-m} = 2NL$$

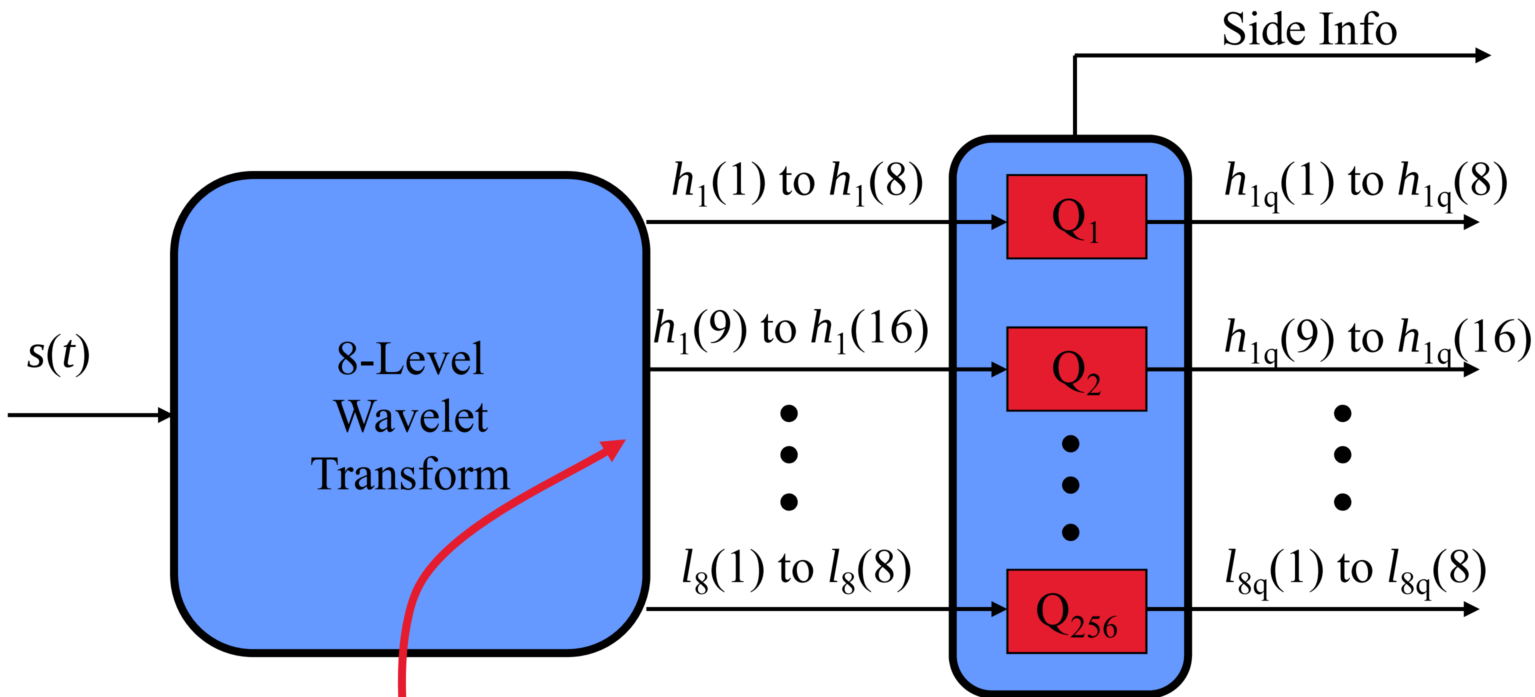
Next

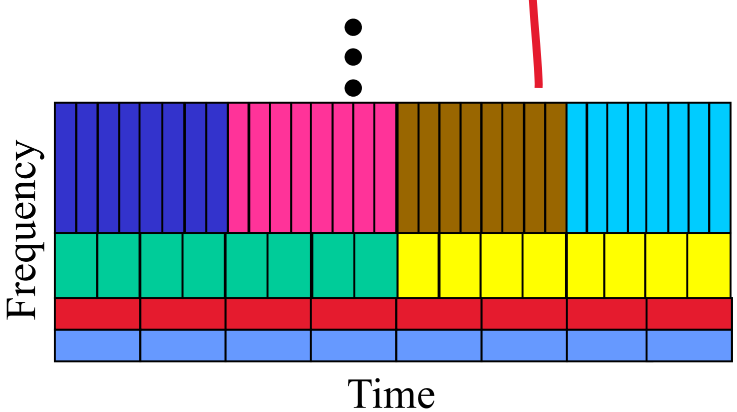
So, What's It Good For?

- signal/image data compression
- computer vision
- enhancing noisy signals
- sonar/radar processing
- biomedical signal processing
- digital communications
- studying turbulance
- geophysical signal processing
- music synthesis

etc., etc., etc.!!!

WT-Based Compression Example





- Bits allocated to quantizers to minimize MSE
- Then allocations less than B_{min} are set to zero
  - » Eliminates negligible cells
- Side info sent to describe allocations

The WT is VERY good at efficiently representing lines and edges

One application is to the compression of fingerprint images:

- FBI uses the WT for compression of its fingerprint images
- Can achieve a 26:1 compression ratio with little degradation
- Avoids the "blocking effects" of JPEG

Original Fingerprint Image   Decoded Fingerprint Image after **26:1 CR**

From http://www.amara.com/IEEEwave/IW_fbi.html

See also http://www.c3.lanl.gov/~brislawn/FBI/FBI.html

Next

Summary

The Wavelet transform provides a means to "see" the time-frequency structure of a signal:

- The WT consists of the coefficients of a signal expansion
  - the basis functions correspond to t-f cells

- The t-f cells adjust their shape to cover the same number of cycles
  - short and wide at low freequencies
  - tall and narrow at high frequencies

- The representation can be easily computed from signal samples
  - simple cascaded filter bank
  - computational complexity is $O(N)$; lower order than the FFT

- The representation is non-redundant (orthogonal)
  - good for compression

- Statistical methods have been developed for de-noising
  - work best when signal is concentrated in WT domain

Papers:

Overview Tutorials

A. Graps, “An Introduction to Wavelets,” *IEEE Computational Science and Engineering*, Summer 1995, pp. 50 - 61.

A. Bruce, D. Donoho, and H. Gao, “Wavelet Analysis,” *IEEE Spectrum*, Oct. 1996, pp. 26 - 35.

M. Cody, “The Fast Wavelet Transform,” *Dr. Dobbs Journal*, April 1992, pp. 16 - 28.
< Code Listing on pp. 100 - 101 >

P. Bentley and J. McDonnell, “Wavelet Transforms: An Introduction,” *Electronics and Communication Engineering Journal*, August 1994, pp. 175 - 186.

Technical Tutorials

O. Rioul and M. Vetterli, “Wavelets and Signal Processing,” *IEEE Signal Processing Magazine*, Oct. 1991, pp. 14 - 38.

A. Cohen and J. Kovacevic, “Wavelets: The Mathematical Background,” *Proceedings of the IEEE*, April 1996, pp. 514 - 522.

N. Hess-Nielsen and M. V. Wickerhauser, “Wavelets and Time-Frequency Analysis,” *Proceedings of the IEEE*, April 1996, pp. 523 - 540.

Books:

The ***first book*** listed gives a nice, gentle overview of wavelets; it is good for technical folks who want to know more but don't have the time to slog through more technical tomes.

The ***second book*** is intended for statisticians, but gives one of the nicest concise treatments I've seen of the mathematical theory of wavelets; it also covers denoising.

The ***other books*** assume a background in standard DSP topics.

B. Burke Hubbard, *The World According to Wavelets*, A. K. Peters, 1995.

R. Todd Ogden, *Essential Wavelets for Statistical Applications and Data Analysis*, Birkhauser, 1997.

M. Vetterli and J. Kovacevic, *Wavelets and Subband Coding*, Prentice-Hall, 1995.

G. Strang and T. Nguyen, *Wavelets and Filter Banks*, Wellesley-Cambridge Press, 1996.

A. Akansu and R. Haddad, *Multiresolution Signal Decompositions: Transforms, Subbands, and Wavelets*, Academic Press.

End