BINGHAMTON UNIVERSITY
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EECE 301
Signals & Systems
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Note Set #19

- D-T Signals: DTFT Details

DTFT Details

What we saw: That the conceptual CTFT inside the DAC can also be computed from the samples… we called that thing the DTFT.

Define the DTFT:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\underbrace{\Omega n}}$$

DT frequency in rad/sample

rad = (rad/sample)× "sample"

Compare to CTFT:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\underbrace{\omega t}} dt$$

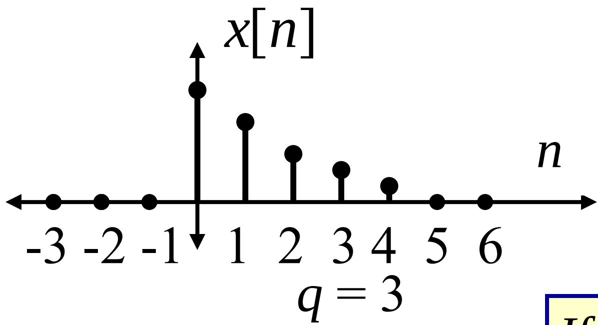
In rad/sec

radians

Very similar structure… so we should expect similar properties!!!

Example of Analytically Computing the DTFT

With your brain, not a computer



$$x[n] = \begin{cases} 0, & n < 0 \\ a^n, & 0 \le n \le q \\ 0, & n > q \end{cases}$$

If $|a| < 1, \quad x[n]$ decays

If $|a| > 1, \quad x[n]$ "explodes"

If $a < 0, \quad x[n]$ oscillates

Given this signal model, find the DTFT.

By definition: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = \sum_{n=0}^{q} a^n e^{-j\Omega n} = \sum_{n=0}^{q} (ae^{-j\Omega})^n$

$$X(\Omega) = \frac{1 - \left(ae^{-j\Omega}\right)^{q+1}}{1 - ae^{-j\Omega}}$$

General Form for Geometric Sum:

$$\sum_{n=q_1}^{q_2} r^n = \frac{r^{q_1} - r^{q_2+1}}{1 - r}$$

Characteristics of DTFT

1.Periodicity of $X(\Omega)$

$X(\Omega)$ is a periodic function of Ω with period of 2π

$$\Rightarrow X(\Omega + 2\pi) = X(\Omega)$$

Recall pictures in notes of "DTFT Intro":

$\Rightarrow |X(\Omega)|$ is periodic with period 2π

$\angle X(\Omega)$ is periodic with period 2π

Note: the CTFT does not have this property

2. $X(\Omega)$ is complex valued (in general)

$$X(\Omega) = \sum_n x[n]\underbrace{e^{-j\Omega n}}_{\text{complex}}$$

Usually think of $X(\Omega)$ in polar form:

$$X(\Omega) = \underbrace{|X(\Omega)|}_{\text{magnitude}} e^{j \overbrace{\angle X(\Omega)}^{\text{phase}}}$$

Same as CTFT

3. Symmetry

If $x[n]$ is real-valued, then:

$$|X(-\Omega)| = |X(\Omega)|$$ (even symmetry)

$$\angle X(-\Omega) = -\angle X(\Omega)$$ (odd symmetry)

Same as CTFT

Inverse DTFT

Q: Given $X(\Omega)$ can we find the corresponding $x[n]$?

A: Yes!!

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{jn\Omega} d\Omega$$

We can integrate instead over any interval of length 2π

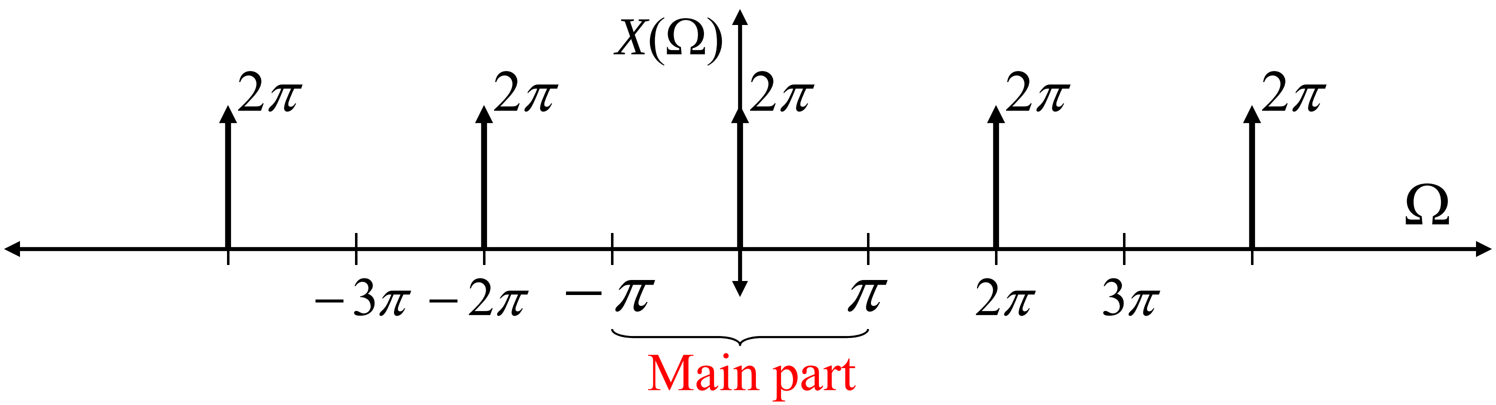
…because the DTFT is periodic with period 2π

Generalized DTFT

Periodic D-T signals have DTFT's that contain delta functions

Example: $x[n]=1,\ \forall n \quad \leftrightarrow \quad X(\Omega)=\begin{cases} 2\pi\delta(\Omega), & -\pi<\Omega<\pi \\ periodic, & elsewhere \end{cases}$

With a period of 2π



Another way of writing this is:

$$X(\Omega)=2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-k2\pi)$$

How do we derive the result? Work backwards!

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{jn\Omega}d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\Omega)e^{jn\Omega}d\Omega$$

Sifting property

$$= e^{jn\cdot 0}$$

$$= 1$$

Transform Pairs: Like for the CTFT, there is a table of common pairs (See Web)

Be familiar with them

Compare and contrast them with the table
Of common CTFT's

DTFT Table

Time Signal	DTFT
$1, \quad -\infty < n < \infty$	$2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$
$\mathrm{sgn}[n] = \begin{cases} -1, & \ldots, -3, -2, -1 \\ 1, & 0, 1, 2, \cdots \end{cases}$	$\frac{2}{1 - e^{-j\Omega}}$
$u[n]$	$\frac{1}{1 - e^{-j\Omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$
$\delta[n]$	$1, \quad -\infty < \Omega < \infty$
$\delta[n-q], \quad q = \pm 1, \pm 2, \pm 3, \ldots$	$e^{-jq\Omega}, \quad q = \pm 1, \pm 2, \pm 3, \ldots$
$a^n u[n], \quad \lvert a \rvert < 1$	$\frac{1}{1 - a e^{-j\Omega}}, \quad \lvert a \rvert < 1$
$e^{j\Omega_o n}, \quad \Omega_o$ real	$2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_o - 2\pi k), \quad \Omega_o$ real
$p_q[n] = \begin{cases} 1, & n = -q, -q+1, \ldots, -1, 0, 1, \ldots q \\ 0, & otherwise \end{cases}$	$\frac{\sin[(q + \frac{1}{2})\Omega]}{\sin(\Omega/2)}$
$\frac{B}{\pi}\mathrm{sinc}[\frac{B}{\pi} n]$	$\sum_{k=-\infty}^{\infty} p_{2B}(\Omega + 2\pi k)$
$\cos(\Omega_o n)$	$\pi \sum_{k=-\infty}^{\infty} [\delta(\Omega + \Omega_o - 2\pi k) + \delta(\Omega - \Omega_o - 2\pi k)]$
$\cos(\Omega_o n + \theta)$	$\pi \sum_{k=-\infty}^{\infty} [e^{-j\theta}\delta(\Omega + \Omega_o - 2\pi k) + e^{j\theta}\delta(\Omega - \Omega_o - 2\pi k)]$
$\sin(\Omega_o n)$	$j\pi \sum_{k=-\infty}^{\infty} [\delta(\Omega + \Omega_o - 2\pi k) - \delta(\Omega - \Omega_o - 2\pi k)]$
$\sin(\Omega_o n + \theta)$	$j\pi \sum_{k=-\infty}^{\infty} [e^{-j\theta}\delta(\Omega + \Omega_o - 2\pi k) - e^{j\theta}\delta(\Omega - \Omega_o - 2\pi k)]$

Fourier Transform Table

Time Signal	Fourier Transform
$1, \quad -\infty < t < \infty$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$1/j\omega$
$u(t)$	$\pi\delta(\omega) + 1/j\omega$
$\delta(t)$	$1, \quad -\infty < \omega < \infty$
$\delta(t - c), \quad c$ real	$e^{-j\omega c}, \quad c$ real
$e^{-bt}u(t), \quad b > 0$	$\frac{1}{j\omega + b}, \quad b > 0$
$e^{j\omega_o t}, \quad \omega_o$ real	$2\pi\delta(\omega - \omega_o), \quad \omega_o$ real
$p_\tau(t)$	$\tau\,\mathrm{sinc}[\tau\omega/2\pi]$
$\tau\,\mathrm{sinc}[\tau t/2\pi]$	$2\pi p_\tau(\omega)$
$\left[1 - \frac{2\lvert t \rvert}{\tau}\right] p_\tau(t)$	$\frac{\tau}{2}\mathrm{sinc}^2[\tau\omega/4\pi]$
$\frac{\tau}{2}\mathrm{sinc}^2[\tau t/4\pi]$	$2\pi\left[1 - \frac{2\lvert\omega\rvert}{\tau}\right] p_\tau(\omega)$
$\cos(\omega_o t)$	$\pi[\delta(\omega + \omega_o) + \delta(\omega - \omega_o)]$
$\cos(\omega_o t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_o) + e^{j\theta}\delta(\omega - \omega_o)]$
$\sin(\omega_o t)$	$j\pi[\delta(\omega + \omega_o) - \delta(\omega - \omega_o)]$
$\sin(\omega_o t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_o) - e^{j\theta}\delta(\omega - \omega_o)]$

DTFT of a Rectangular Pulse

Define: D-T pulse as $p_q[n] = \begin{cases} 1, & n = -q, \ldots, -1, 0, 1, \ldots, q \\ 0, & otherwise \end{cases}$

Subscript tells how far "left and right"

Use "Geometric Sum" Result…

So, by DTFT definition: $P_q(\Omega) = \sum_{n=-q}^{q} e^{-jn\Omega}$

$$P_q(\Omega) = \frac{e^{jq\Omega} - e^{-j(q+1)\Omega}}{1 - e^{-j\Omega}} = \frac{\sin\{(q+1/2)\Omega\}}{\sin\{\Omega/2\}}$$

See book for details

Properties of the DTFT (See table provided)

Like for the CTFT, there are many properties for the DTFT. Most are identical to those for the CTFT!!

But Note: "Summation Property" replaces Integration

There is no "Differentiation Property"

Most important ones:

-Time shift

-Multiplication by sinusoid… Three "flavors"

-Convolution in the time domain

-Parseval's Theorem

Compare and contrast these with the table of CTFT properties

Comparing Properties of DTFT & CTFT

DTFT Properties

Property Name	Property	
Linearity	$ax[n]+bv[n]$	$aX(\Omega)+bV(\Omega)$
Time Shift	$x[n-q]$, q any integer	$e^{-jq\Omega}X(\Omega)$, q any integer
Time Scaling	$x(at)$, $a \neq 0$	$\frac{1}{a}X(\Omega/a)$, $a \neq 0$
Time Reversal	$x[-n]$	$X(-\Omega)$ $\overline{X(\Omega)}$ if $x[n]$ is real
Multiply by n	$nx[n]$	$j\frac{d}{d\Omega}X(\Omega)$
Multiply by Complex Exponential	$e^{j\Omega_o n}x[n]$, Ω_o real	$X(\Omega-\Omega_o)$, Ω_o real
Multiply by Sine	$\sin(\Omega_o n)x[n]$	$\frac{j}{2}[X(\Omega+\Omega_o)-X(\Omega-\Omega_o)]$
Multiply by Cosine	$\cos(\Omega_o n)x[n]$	$\frac{1}{2}[X(\Omega+\Omega_o)+X(\Omega-\Omega_o)]$
Summation	$\sum_{i=-\infty}^{n}x[i]$	$\frac{1}{1-e^{-j\Omega}}X(\Omega)+\pi\sum_{k=-\infty}^{\infty}X(0)\delta(\Omega-2\pi k)$
Convolution in Time	$x[n]*h[n]$	$X(\Omega)H(\Omega)$
Multiplication in Time	$x[n]w[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\Omega-\lambda)W(\lambda)d\lambda$ (conv.)
Parseval's Theorem (General)	$\sum_{n=-\infty}^{\infty}x[n]\overline{v[n]}=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\Omega)\overline{V(\Omega)}d\Omega$	
Parseval's Theorem (Energy)	$\sum_{n=-\infty}^{\infty}x^2[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(\Omega)\rvert^2 d\Omega$ if $x(t)$ is real $\sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(\Omega)\rvert^2 d\Omega$	
Using CTFT Table to find Inverse of a DTFT $X(\Omega)$: $x[n]=??$	Form $\Gamma(\omega)=X(\omega)p_{2\pi}(\omega)$ and look up $\gamma(t)\leftrightarrow\Gamma(\omega)$ Then get $x[n]=\gamma(t)\rvert_{t=n}$	

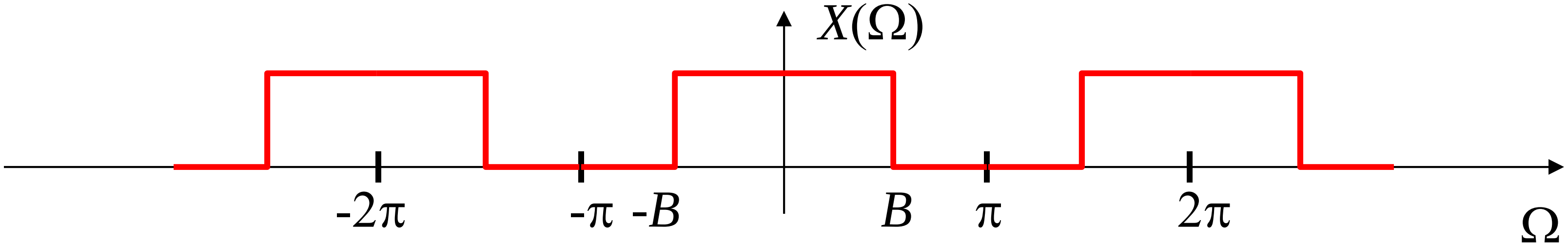
Fourier Transform Properties

Property Name	Property	
Linearity	$ax(t)+bv(t)$	$aX(\omega)+bV(\omega)$
Time Shift	$x(t-c)$	$e^{-j\omega c}X(\omega)$
Time Scaling	$x(at)$, $a \neq 0$	$\frac{1}{a}X(\omega/a)$, $a \neq 0$
Time Reversal	$x(-t)$	$X(-\omega)$ $\overline{X(\omega)}$ if $x(t)$ is real
Multiply by t^n	$t^n x(t)$, $n=1,2,3,\ldots$	$j^n\frac{d^n}{d\omega^n}X(\omega)$, $n=1,2,3,\ldots$
Multiply by Complex Exponential	$e^{j\omega_o t}x(t)$, ω_o real	$X(\omega-\omega_o)$, ω_o real
Multiply by Sine	$\sin(\omega_o t)x(t)$	$\frac{j}{2}[X(\omega+\omega_o)-X(\omega-\omega_o)]$
Multiply by Cosine	$\cos(\omega_o t)x(t)$	$\frac{1}{2}[X(\omega+\omega_o)+X(\omega-\omega_o)]$
Time Differentiation	$\frac{d^n}{dt^n}x(t)$, $n=1,2,3,\ldots$	$(j\omega)^n X(\omega)$, $n=1,2,3,\ldots$
Time Integration	$\int_{-\infty}^{t}x(\lambda)d\lambda$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$
Convolution in Time	$x(t)*h(t)$	$X(\omega)H(\omega)$
Multiplication in Time	$x(t)w(t)$	$\frac{1}{2\pi}X(\omega)*W(\omega)$
Parseval's Theorem (General)	$\int_{-\infty}^{\infty}x(t)\overline{v(t)}dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)\overline{V(\omega)}d\omega$	
Parseval's Theorem (Energy)	$\int_{-\infty}^{\infty}x^2(t)dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2 d\omega$ if $x(t)$ is real $\int_{-\infty}^{\infty}\lvert x(t)\rvert^2 dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2 d\omega$	
Duality: If $x(t) \leftrightarrow X(\omega)$	$X(t)$	$2\pi x(-\omega)$

This one has no equivalent on CTFT Properties Table… See next example

It provides a way to use a CTFT table to find DTFT pairs… here is an example

Example: Finding a DTFT pair from a CTFT pair

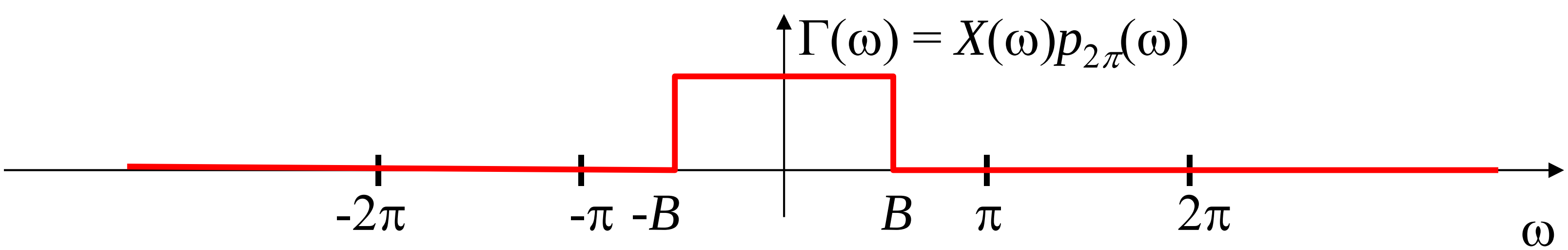


Say we are given this DTFT and want to invert it…

The four steps for using "Relationship to Inverse CTFT" property are:

1. Truncate the DTFT X(Ω) to the -π to π range and set it to zero elsewhere
2. Then treat the resulting function as a function of ω… call this Γ(ω)

$$\Gamma(\omega) = X(\omega)p_{2\pi}(\omega)$$



3. Find the inverse CTFT of $\Gamma(\omega)$ from a CTFT table, call it $\gamma(t)$

From CTFT table:

$$\gamma(t) = \frac{B}{\pi}\text{sinc}\left(\frac{B}{\pi}t\right)$$

4. Get the $x[n]$ by replacing t by n in $\gamma(t)$

$$x[n] = \gamma(t)\big|_{t=n} = \frac{B}{\pi}\text{sinc}\left(\frac{B}{\pi}n\right)$$

Example of DTFT of sinusoid

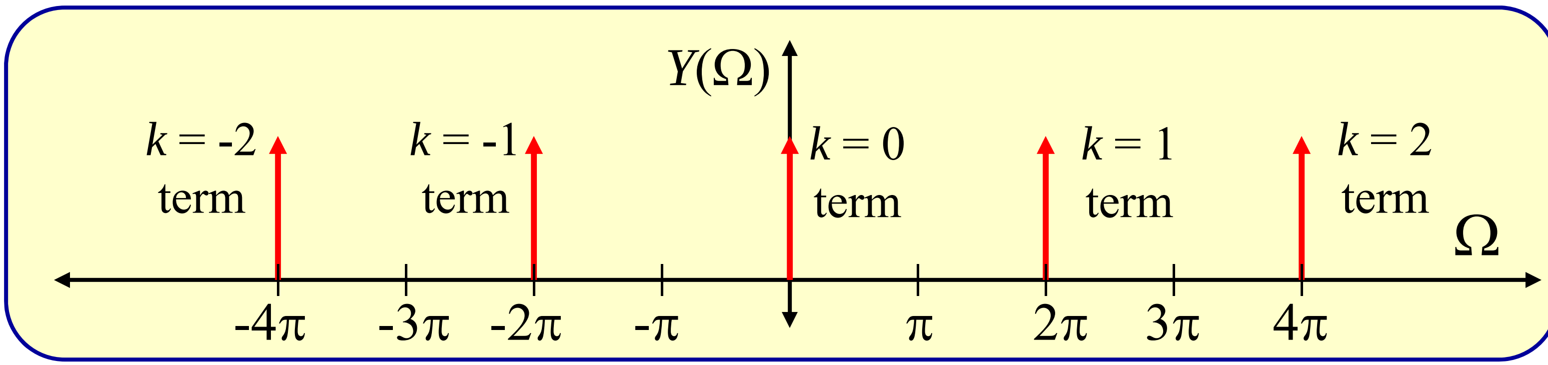
$$x[n] = \cos(\Omega_0 n) \quad \leftrightarrow \quad X(\Omega) = ?$$

Note that: $x[n] = 1 \times \cos(\Omega_0 n)$

So… use the "mult. by sinusoid" property

$\triangleq y[n] = 1$ → From DTFT Table →

$$Y(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$



Another way of writing this:

$$Y(\Omega) = \begin{cases} 2\pi\delta(\Omega), & -\pi < \Omega < \pi \\ 2\pi - \text{periodic elsewhere} \end{cases}$$

Recall: $x[n] = 1 \times \cos(\Omega_0 n)$ so we can use the "mult. by sinusoid" result

$$\Rightarrow X(\Omega) = \frac{1}{2}\left[Y(\Omega + \Omega_0) + Y(\Omega - \Omega_0)\right]$$

"mult. by sinusoid" property says we shift up & down by Ω_0

Using the second form for $Y(\Omega)$ gives:

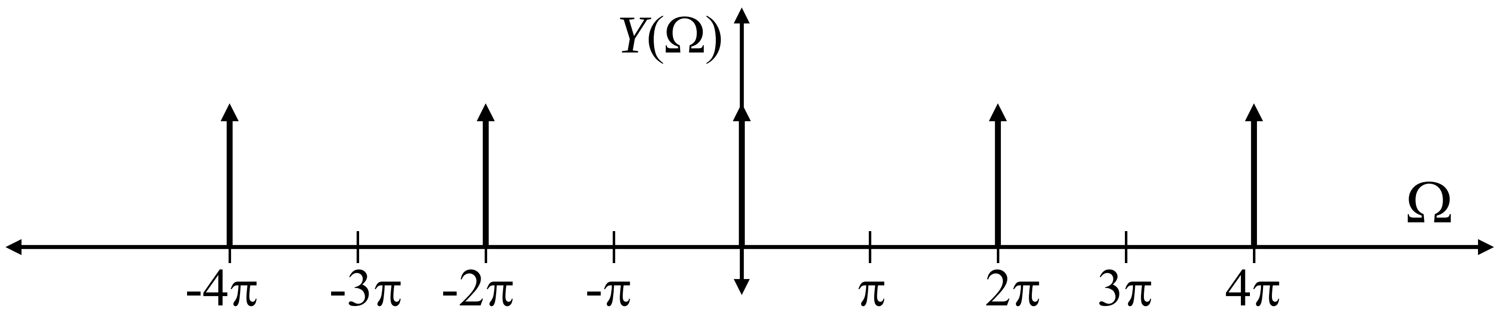
$$X(\Omega) = \begin{cases} \pi\left[\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)\right], & -\pi < \Omega < \pi \\ 2\pi - \text{periodic elsewhere} \end{cases}$$

Or…using the first form for $Y(\Omega)$ gives:

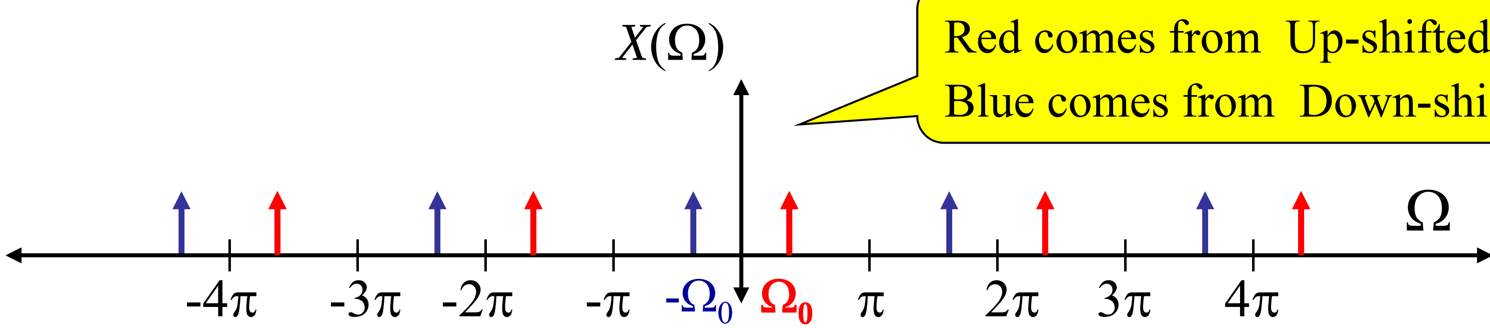
$$Y(\Omega) = \pi \sum_{k=-\infty}^{\infty}\left[\delta(\Omega + \Omega_0 - 2\pi k) + \delta(\Omega - \Omega_0 - 2\pi k)\right]$$

To see this graphically:

$$Y(\Omega) = \begin{cases} 2\pi\delta(\Omega), & -\pi < \Omega < \pi \\ 2\pi - \text{periodic elsewhere} \end{cases}$$



$$X(\Omega) = \begin{cases} \pi[\delta(\Omega+\Omega_0)+\delta(\Omega-\Omega_0)], & -\pi < \Omega < \pi \\ 2\pi - \textit{periodic elsewhere} \end{cases}$$



Red comes from Up-shifted $Y(\Omega)$

Blue comes from Down-shifted $Y(\Omega)$

Comment on Some DTFT Forms on the Table

The last four entries on the DTFT Pairs Table are:

$\cos(\Omega_o n)$	$\pi \sum_{k=-\infty}^{\infty} \left[\delta(\Omega+\Omega_o-2\pi k)+\delta(\Omega-\Omega_o-2\pi k)\right]$
$\cos(\Omega_o n+\theta)$	$\pi \sum_{k=-\infty}^{\infty} \left[e^{-j\theta}\delta(\Omega+\Omega_o-2\pi k)+e^{j\theta}\delta(\Omega-\Omega_o-2\pi k)\right]$
$\sin(\Omega_o n)$	$j\pi \sum_{k=-\infty}^{\infty} \left[\delta(\Omega+\Omega_o-2\pi k)-\delta(\Omega-\Omega_o-2\pi k)\right]$
$\sin(\Omega_o n+\theta)$	$j\pi \sum_{k=-\infty}^{\infty} \left[e^{-j\theta}\delta(\Omega+\Omega_o-2\pi k)-e^{j\theta}\delta(\Omega-\Omega_o-2\pi k)\right]$

- Note that each of them has a summation… where the summation just adds in terms that are shifted by 2π
- Note that because of this shift, only the $k = 0$ term lies between $-\pi$ and π
- Thus… we could more simply state these by writing only the $k = 0$ term and stating that the result is 2π-periodic elsewhere… like this:

$$\frac{B}{\pi}\operatorname{sinc}\left[\frac{B}{\pi}n\right] \longleftrightarrow \begin{cases} p_{2B}(\Omega), -\pi \le \Omega \le \pi \\ 2\pi\text{-periodic elsewhere} \end{cases}$$