

## Wavelet Transforms and Signal Ambiguity Functions

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### ABSTRACT

Classical results from group representation theory are used to gain insight into important properties of narrowband and wideband ambiguity functions and wavelet transforms. Wideband ambiguity functions are essentially affine wavelet transforms and narrowband ambiguity functions can be considered to be Heisenberg wavelet transforms. Important invariance properties of the ambiguity functions are consequences of the group representation theory. This paper presents examples of derivations of wideband ambiguity function properties.

### INTRODUCTION

There has been a recent explosion of research interest in continuous and discrete wavelet transforms [1, 2, 3, 4, 5, 6, 7]. The wavelet transforms can be considered to be time-frequency localization operators that treat the time-frequency plane (phase space or in terms of Gabor, the information plane) as one geometric whole rather than as two separate spaces [5]. Localization of signals in the time  $\times$  frequency plane is the essence of the signal ambiguity function problem. In the case of narrowband ambiguity functions, one computes the inner product of a received signal with a time delayed and Doppler shifted version of the reference signal  $g(t)$ .

All the time delayed and Doppler shifted signals can be obtained from the elementary signal  $g(t)$  by a unitary operator that represents a Heisenberg group element. Thus, the narrowband ambiguity function can be interpreted as the generalized Gabor transform of the received signal with respect to the reference signal  $g(t)$ .

Gabor originally used  $g(t) = e^{-\tau t^2}$  because it has the minimum possible product of rms duration and bandwidth [9]. For this reason also, Gabor transforms have fundamental significance in physics and information theory. Later, Montgomery and Reed [10] showed that the Gabor transform is well defined for any reference signal  $g(t) \in L^2(\mathcal{R})$ ; this result is easily verified using the theory of group representations [1]. To emphasize the connection to the Heisenberg group, we will refer to the Gabor transform as the *Heisenberg wavelet transform*.

In the case of wideband ambiguity functions, one computes the inner product of a received signal with a time delayed and dilated version of the reference signal  $g(t)$ . These time delayed and dilated signals can be obtained from  $g(t)$  by a unitary operator that represents an affine group element. Hence wideband cross-ambiguity functions

are *affine wavelet* transforms. Auto-ambiguity functions are special cases of cross-ambiguity functions.

Group representation theory provides a unified framework for the study of narrowband and wideband ambiguity functions, as well as Heisenberg and affine wavelet transforms. The important concept is that both narrowband and wideband ambiguity functions are coefficients of the unitary representations of their respective groups. Various well known invariance properties of narrowband and wideband ambiguity functions follow easily from this fact. We demonstrate the latter by several examples.

Group representation theory also helps to explain the essential differences between narrowband and wideband ambiguity functions and related wavelet transforms. In particular it gives us new insight into the ambiguity volume conservation and admissible signals. Ambiguity volume conservation is a property of practical significance for the synthesis of realizable signals with desired ambiguity properties.

The objective of this article, an up-dated version of our earlier paper [21], is to provide a unified group theoretic treatment of ambiguity functions and wavelet theory. Additional results that are presented here are group theoretic derivations of ambiguity function invariance properties.

The work makes use of the results of considerable recent literature on this topic [1, 2, 3, 4, 5, 6, 7]. Heil and Walnut [1], for instance, present a good up-to-date overview of wavelet transform research in a paper highly recommended to readers interested in this fascinating topic. Group theoretic aspects of narrowband ambiguity functions have been investigated also by Auslander and Tolimieri [11] and by Schempp [12, 13, 14]. Our discussion emphasizes the connection between affine wavelet transforms and wideband ambiguity functions, with the key integral formulas to be discussed in this paper being:

Narrowband Ambiguity Function:

$$H(f, g; x, y) = \int_{-\infty}^{\infty} f(t)[g(t-x)]^* e^{-2\pi i y t} dt$$

Wideband Ambiguity Function:

$$W(f, g; s, \tau) = s^{-1/2} \int_{-\infty}^{\infty} f(t)[g(st + \tau)]^* dt$$

SOME MATHEMATICAL BACKGROUND

Basic knowledge of group theory as presented in [15], and knowledge of Hilbert space in [16] and [17] will be assumed. However, the definition of a group will be reviewed.

A group  $G$  is a set  $G$  with a map  $*$ :  $G \times G \rightarrow G$  denoted by  $(x, y) \mapsto x * y$  having the following properties:

1. Closure:  $x * y \in G$  for all  $x, y \in G$ .
2. Associativity:  $(x * y) * z = x * (y * z)$ .
3. There is an element  $e \in G$ , to be called the *identity*, such that  $e * x = x * e = x$  for all  $x \in G$ .
4. For each  $x \in G$ , there is an element,  $x^{-1}$ , to be called the *inverse of  $x$*  such that:  $x * x^{-1} = x^{-1} * x = e$ .

If in addition,  $x * y = y * x$  for all  $x, y \in G$ , the group is said to be *commutative*.

When the operation  $*$  is totally clear, the symbol  $*$  will be omitted:  $xy$  instead of  $x * y$ .

A subset  $S$  of group  $G$  is a subgroup of  $G$  if it is a group under the same operation as  $G$ .

The group  $G$  may have additional structure. For instance, if  $G$  is a topological space and  $*$  is continuous with respect to this topology,  $G$  will be a topological group. Furthermore, if  $G$  has a differentiable structure,  $G$  will be a differentiable manifold, and called a *Lie Group*.

A differentiable manifold for our purpose is a set which "looks like" Euclidean space. For example,  $\mathcal{R}^n$  is one since it is Euclidean space. On Euclidean space, we can choose different parametrizations; however, the change of coordinates must be differentiable. Rectangular coordinates and polar coordinates are familiar examples of such parametrizations of  $\mathcal{R}^2$ . We recall that change of coordinates must also be invertible locally, that is, that the Jacobian of the transformation be invertible.

There is a vast amount of theory of Lie groups. For purposes of this paper, we will concentrate on a special category of groups called the *linear groups*. These are groups which can be realized as groups of linear transformations of a vector space, or equivalently, as matrix groups.

The set  $GL(n, \mathcal{R})$  of invertible matrices with real coefficients form a group called the *real general linear group*, under the usual matrix multiplication. Indeed, the set  $GL(n, \mathcal{R})$  may be parametrized using  $n^2$  coordinate functions, by fixing a basis in  $\mathcal{R}^n$ . Thus  $GL(n, \mathcal{R})$  "looks like"  $\mathcal{R}^{n^2}$ .

It is a fact that every group of invertible linear transformations of a real  $n$ -dimensional vector space may be seen as a subgroup of  $GL(n, \mathcal{R})$ .

The subgroup of  $GL(2, \mathcal{R})$  consisting of matrices of the form

$$\begin{pmatrix} s & t \\ 0 & 1 \end{pmatrix},$$

with  $s > 0$  and  $t$  any real number, is called the *real affine group*. It will be denoted by  $\mathcal{A}^1$ . Its identity element is the usual identity element of  $GL(2, \mathcal{R})$ , and the inverse of an element is

$$\begin{pmatrix} s^{-1} & -s^{-1}t \\ 0 & 1 \end{pmatrix}$$

Another matrix group of particular interest to this paper is the so-called Heisenberg group,  $\mathcal{H}$ , the group of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

It can be verified that the identity and inverse of this set belongs to  $\mathcal{H}$ , and that a parametrization of the Heisenberg group is the one that associates a matrix with its three superdiagonal entries. One sees, then, that  $\mathcal{H}$  "looks like"  $\mathcal{R}^3$ .

The set of operators on a Hilbert space that preserve its inner product forms a group under composition, called the *Unitary Group* of the Hilbert space involved. Specifically,  $L^2(\mathcal{R})$  will be our Hilbert space under consideration.

It is possible to define measures on topological groups and hence Lie groups, such as the Heisenberg group and the affine group. These measures are compatible with the group structure in the sense that they remain unchanged if the set is translated by a multiplication by a group element. Since groups are not commutative in general, there are both *left invariant* and *right invariant* measures. If  $f$  is a measurable function on a group  $G$ ,  $\mu$  is left-invariant, if

$$\int_G f(g_0 * g) d\mu(g) = \int_G f(g) d\mu(g),$$

for all  $g_0 \in G$ . The right invariant measure is defined similarly. If an invariant measure, left or right, exists for a group, it is well known that it must necessarily be unique up to multiplication by a constant. Finally, if the left and right invariant measure coincide, the group in question is said to be *unimodular*.

An example of left-invariant measure is the well known Lebesgue (or Riemann) measure. The group involved is the group of real numbers under addition. Certainly,

$$\int_{-\infty}^{\infty} f(x + y) dx = \int_{-\infty}^{\infty} f(x) dx,$$

for all  $y \in \mathcal{R}$ . This group is unimodular since the real numbers are commutative under addition.

A *unitary representation*  $U$  of a group  $G$  on a Hilbert space  $H$  is a mapping assigning to each group element a unitary operator  $U(x)$  on  $H$ , such that for any  $x, y \in G$ ,

$$U(x * y) = U(x) \circ U(y).$$

(Recall that  $U$  is a unitary operator on  $H$  if, for all  $v, w \in H$ ,  $\langle Uv, Uw \rangle = \langle v, w \rangle$ ).

A subspace  $S$  of  $H$  is *invariant* under  $U$  if for any  $v \in S$ ,  $U(x) \cdot v \in S$  for all  $x \in G$ . A representation  $U$  is said to be *irreducible* if the only invariant subspaces of  $H$  are the zero subspace and  $H$  itself.

An irreducible unitary representation  $U$  of a group  $G$  on a Hilbert space  $H$  is said to be *square integrable* if there is some  $g \in H$  such that:

$$\int_G |\langle g, U(x)g \rangle|^2 d\mu(x) < \infty,$$

where  $\mu$  is the left invariant measure of  $G$ . Such a  $g$  is said to be an *admissible vector*.

Square integrable representations enjoy the following property, described by a generalization of the Frobenius-Schur-Godement theorem [13]: If  $U$  is a square integrable representation of a group  $G$  acting on a Hilbert space  $H$ , there exists a unique self-adjoint positive operator  $Q$  such that:

- i) The set of admissible vectors coincide with the domain of  $Q$ .
- ii) Let  $g_1$  and  $g_2$  be two admissible vectors. Let  $f_1$  and  $f_2$  be any vectors in  $H$ , then

$$\int_G \langle f_1, U(x)g_1 \rangle \langle f_2, U(x)g_2 \rangle^* dx = \langle Qg_2, Qg_1 \rangle \langle f_1, f_2 \rangle.$$

Finally, if  $G$  is unimodular, then  $Q$  is a multiple of the identity operator; hence the domain of  $Q$  will be all of  $H$ , that is, all of  $H$  constitutes the set of admissible vectors for the representation.

The above theorem is proved in detail in [2]. It will play a major role in determining the volume under the ambiguity surface.

The *coefficient* of a continuous unitary group representation  $U$  of  $G$  on a Hilbert space  $H$  with respect to the ordered pair  $(f, g)$ , both  $f$  and  $g$  in  $H$ , is a mapping  $C_{U,f,g} : G \rightarrow \mathcal{R}$  such that

$$x \mapsto \langle f, U(x)g \rangle.$$

This coefficient is a generalization of the Fourier coefficients, and will be useful in rephrasing well known concepts later.

For notational convenience, we define the following operators on  $L^2(\mathcal{R})$ :

- Translation:  $T_a f(x) = f(x - a)$ ,  $a \in \mathcal{R}$ .
- Dilation:  $D_k f(x) = \sqrt{k} f(kx)$ ,  $k > 0$ .
- Modulation:  $E_a f(x) = e^{2\pi i a x} f(x)$ ,  $a \in \mathcal{R}$ .
- Multiplication:  $M_a f(x) = e^{2\pi i a} f(x)$ ,  $a \in \mathcal{R}$ .

These operators are easily seen to be unitary.

Further information concerning group representations may be found in [18].

#### THE WIDE AND NARROW BAND AMBIGUITY FUNCTIONS

It is the purpose of this section to show that the wide and narrow band ambiguity functions are the coefficients of the affine and Heisenberg group respectively. This essentially is the vital connection between the functions and group theory.

Consider the Heisenberg group,  $\mathcal{H}$ . It has a unitary representation  $U$  on  $L^2(\mathcal{R})$  as follows:  $U(x, y, z)$  is the unitary operator corresponding to  $(x, y, z)$  in  $\mathcal{H}$  such that:

$$\begin{aligned} U(x, y, z)f(t) &= M_z T_x E_y f(t) \\ &= z e^{2\pi i y(t-x)} f(t-x). \end{aligned}$$

That this unitary representation of  $\mathcal{H}$  is unique in a certain sense and irreducible is the content of the Stone-Von Neuman Theorem [7].

When we apply the definition of coefficient to  $U$  on  $L^2(\mathcal{R})$  with respect to  $(f, g)$ , we obtain

$$\begin{aligned} C_{U,f,g} &= \langle f, U(x, y, z)g \rangle \\ &= \int_{-\infty}^{\infty} f(t) [M_z T_x E_y g(t)]^* dt \\ &= \int_{-\infty}^{\infty} f(t) [z g(t-x)]^* e^{-2\pi i y(t-x)} dt. \end{aligned}$$

When we set  $z = 1$ , we find that the coefficient takes the form of the narrow band cross ambiguity function. Of course, letting  $f = g$  leads to the narrow band ambiguity function, a connection with the Heisenberg group which allows us to discover many properties.

One of the more important of these properties is the volume conservation property of the narrow band ambiguity functions. To derive this abstractly, one must prove that the representation  $U$  of  $\mathcal{H}$  is square integrable. However, since the parametrization in the previous section does not lead to a square integrable representation, it is necessary to display a new parametrization as follows:

$$(x, y, z) \mapsto (x, y, e^{2\pi i x}).$$

This parametrization changes the multiplication rule:

$$(x, y, e^{2\pi i x}) * (x', y', e^{2\pi i x'}) = (x + x', y + y', e^{2\pi i(x+x'+xy')}).$$

As one can see by the above, the reparametrization really only consists of exponentiating the third component of each element.

Under this reparametrization, the left and right invariant measure of  $\mathcal{H}$  can be computed to be:  $dx dy dz / 2\pi$ , the proof of which is contained in [1]. This, by the way, shows an example of a unimodular group which is not commutative. By the Frobenius-Schur-Godement theorem, all of  $L^2(\mathcal{R})$  is the set of admissible vectors. An explicit proof is presented in [1].

By the same theorem, substituting appropriately for the case of  $\mathcal{H}$ , then

$$\begin{aligned} \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle f_1, U(x, y, z)g_1 \rangle \langle f_2, U(x, y, z)g_2 \rangle^* dx dy dz / 2\pi \\ = \langle g_2, g_1 \rangle \langle f_1, f_2 \rangle. \end{aligned}$$

If we let  $f = f_i = g_i$ ,  $i = 1, 2$ , then we obtain:

$$\int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\langle f, U(x, y, z)f \rangle|^2 dx dy dz / 2\pi = \|f\|^4.$$

If we further assume that  $\|f\| = 1$ , the above expression simplifies to:

$$\int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\langle f, U(x, y, z)f \rangle|^2 dx dy dz / 2\pi = 1.$$

This is the statement of the conservation of ambiguity volume. Seen in this context, the conservation property is true of any "ambi-

guity function" which is a matrix coefficient of a unimodular group.

The conservation of narrowband ambiguity volume having been proved, it is natural to ask whether a similar conservation law holds for wide band ambiguity functions. Unfortunately, the answer is no, based on group theoretic reasoning.

The Affine group,  $\mathcal{A}^1$  has a unitary representation  $A$  on  $L^2(\mathcal{R})$  as follows:  $A(s, \tau)$  is the unitary operator corresponding to  $(s, \tau)$  in  $\mathcal{A}^1$  of the form:

$$\begin{aligned} A(s, \tau)f(t) &= T_\tau D_s f(t) \\ &= \sqrt{s}f(st + \tau). \end{aligned}$$

As in the case of  $\mathcal{H}$ , we form the coefficient of this representation:

$$\begin{aligned} C_{A,fg} &= \langle f, A(s, \tau)g \rangle \\ &= \int_{-\infty}^{\infty} f(t)[T_\tau D_s g(t)]^* dt \\ &= \int_{-\infty}^{\infty} f(t)[\sqrt{s}g(st + \tau)]^* dt, \end{aligned}$$

which is the wavelet transform of  $f$  with respect to  $g$ .

One immediately sees that this coefficient is no other than the wide band ambiguity function.

The group  $\mathcal{A}^1$  may be reparametrized in the form:

$$(s, \tau) \mapsto (e^{-s}, \tau),$$

where  $s$  can take on all real values. The corresponding left invariant measure can be shown to be  $e^s ds d\tau$  and the right invariant measure is  $ds d\tau$ .  $\mathcal{A}^1$  is thus not a unimodular group. It is proved in [1] that the admissible vectors of the representation above are those  $g$  such that:

$$\int_0^\infty \frac{\hat{g}(\omega)}{\omega} d\omega < \infty,$$

where  $\hat{g}(\omega)$  denotes the Fourier transform of  $g(t)$ .

Letting  $s$  and  $\tau$  vary,  $C_{A,fg}$  traces a surface in space called the *wideband ambiguity surface*. Unlike the narrowband case, the volume under this surface for  $\|f\| = 1$  is *not* constant. Indeed the same calculation proving the square integrability of  $A$  yields:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\langle f, A(s, \tau)g \rangle|^2 ds d\tau = \int_0^\infty |\hat{f}(\omega)|^2 d\omega \int_0^\infty \frac{|\hat{g}(\gamma)|^2}{|\gamma|} d\gamma$$

Letting  $f = g$ , we obtain the volume formula:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\langle f, A(s, \tau)f \rangle|^2 ds d\tau = \int_0^\infty |\hat{f}(\omega)|^2 d\omega \int_0^\infty \frac{|\hat{f}(\gamma)|^2}{|\gamma|} d\gamma$$

which agrees with an independent calculation by Sibul and Titlebaum [19]. Hence wideband ambiguity is not conserved.

Thus some properties of ambiguity functions may be traced back to properties of groups they represent.

## INVARIANCE PROPERTIES

In [20], Altes proved various invariance properties of the wideband ambiguity function. These results can also be viewed as a consequence of the wideband ambiguity function being the coefficient of a unitary representation of the affine group. However, Altes used the formula:

$$W_{f,g} = \int_{-\infty}^{\infty} f(t)g^*[s(t + \tau)]dt,$$

instead of:

$$W_{f,g} = \int_{-\infty}^{\infty} f(t)g^*(st + \tau)dt.$$

Both forms are used in the engineering literature. The second form, however, is a coefficient of the affine group, and so exhibits much symmetry. We will focus on the second form, however it is not difficult to transform it to the first by a change of variable.

As a first example, Altes proved that  $W_{f,g}(s, \tau) = W_{g^*,f}^*(s^{-1}, -\tau/s)$ . This fact can be seen from a group theoretic point of view as follows:

$$W_{f,g}(s, \tau) = \langle f, A(s, \tau)g \rangle.$$

But  $A$  is a unitary representation of the affine group and so the inverse of  $A$  is equal to the adjoint of  $A$  and

$$\begin{aligned} \langle f, A(s, \tau)g \rangle &= \langle [A(s, \tau)]^{-1}f, [A(s, \tau)]^{-1}A(s, \tau)g \rangle \\ &= \langle A(1/s, -\tau/s)f, g \rangle \\ &= \langle g, A(1/s, -\tau/s)f \rangle^* \\ &= W_{g^*,f}^*(1/s, -\tau/s). \end{aligned}$$

It can thus be seen that this proof gives some indication of the symmetries underlying the ambiguity function.

The symmetry exploitation idea may be extended by examining relationships between various subgroups of the affine group. The subgroup of dilations, the matrices of the form

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$$

has a unitary representation  $D$  on  $L^2(\mathcal{R})$ :

$$D_k f(t) = \sqrt{k}f(kt).$$

But it can be shown that  $D_k \circ A(s, \tau) = A(s, \tau) \circ D_k$  by multiplying matrices. Hence the representations of the dilation subgroup and the affine group commute as well. This is the basis for Altes' result: If  $f(t) = k^{1/2}u(kt)$  and  $g(t) = k^{1/2}v(kt)$ , then

$$|W_{g,f}(s, \tau)|^2 = |W_{v,u}(t, s)|^2.$$

Indeed,

$$\begin{aligned} W_{g,f}(s, \tau) &= \langle g, A(s, \tau)f \rangle \\ &= \langle D_k v, A(s, \tau)D_k u \rangle \\ &= \langle D_k v, D_k A(s, \tau)u \rangle \\ &= \langle v, A(s, \tau)u \rangle \\ &= W_{v,u}(s, \tau). \end{aligned}$$

Our final example will illustrate another aspect of group theory. A unitary representation of a Lie group  $G$  can be seen as a map from  $G$  to the unitary group of  $L^2(\mathcal{R})$  possessing nice properties such as differentiability, and preservation of group structures. Technically, such a map is called a *differentiable homomorphism*, although such terminology need not concern us here. The image of this type of map is known to be a subgroup of the unitary group, therefore it is possible to investigate relationships between the unitary representation operators and other unitary operators.

One such operator is the "inversion operator"  $\mathcal{I} : L^2(\mathcal{R}) \rightarrow L^2(\mathcal{R})$  :

$$\mathcal{I}[f(t)] = f(-t).$$

$\mathcal{I}$  is not a representation of the affine group since it is equivalent to a negative dilation. However, we can prove that:

$$A(s, \tau) \circ \mathcal{I} = \mathcal{I} \circ A(s, -\tau).$$

Indeed,

$$\begin{aligned} A(s, \tau) \circ \mathcal{I}[f(t)] &= A(s, \tau)[f(-t)] \\ &= f(-st + \tau) \\ &= f(-(st - \tau)) \\ &= \mathcal{I}f(st - \tau) \\ &= \mathcal{I} \circ A(s, -\tau)[f(t)], \end{aligned}$$

which was to be proven.

This fact can be used to deduce one of Altes' results: If  $f(t) = u(-t)$  and  $g(t) = v(-t)$ , then  $W_{g,f}(s, \tau) = W_{v,u}(s, -\tau)$ . The idea of the proof is similar to the others:

$$\begin{aligned} W_{g,f}(s, \tau) &= \langle g, A(s, \tau)f \rangle \\ &= \langle \mathcal{I}u, A(s, \tau)\mathcal{I}v \rangle \\ &= \langle \mathcal{I}u, \mathcal{I}A(s, -\tau)v \rangle \\ &= \langle u, A(s, -\tau)v \rangle \\ &= W_{u,v}(s, -\tau). \end{aligned}$$

The next to last step follows from the fact that  $\mathcal{I}$  is a unitary operator.

Many other properties of the wide band ambiguity function may be proved using these ideas. In fact, similar properties of the narrow band ambiguity function may be obtained by the fact that it is the coefficient of a representation of the Heisenberg group. Invariance properties of the wide and narrow band ambiguity functions may be discovered and proved by investigating properties of the group elements within the groups to which they are related.

### CONCLUSIONS

Wideband and narrowband ambiguity functions are time  $\times$  frequency plane localization operators, as are corresponding wavelet transforms. Wideband ambiguity functions are affine wavelet transforms and narrowband ambiguity functions are generalized Gabor transforms or Heisenberg wavelet transforms. In this and in a previous paper [21] we have shown that group representation theory provides a unified framework for the study of all above-mentioned time

$\times$  frequency plane (phase space) localization operators. In particular, group representation theory has provided new insight into admissible signals, ambiguity volume conservation, and ambiguity function invariance properties.

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