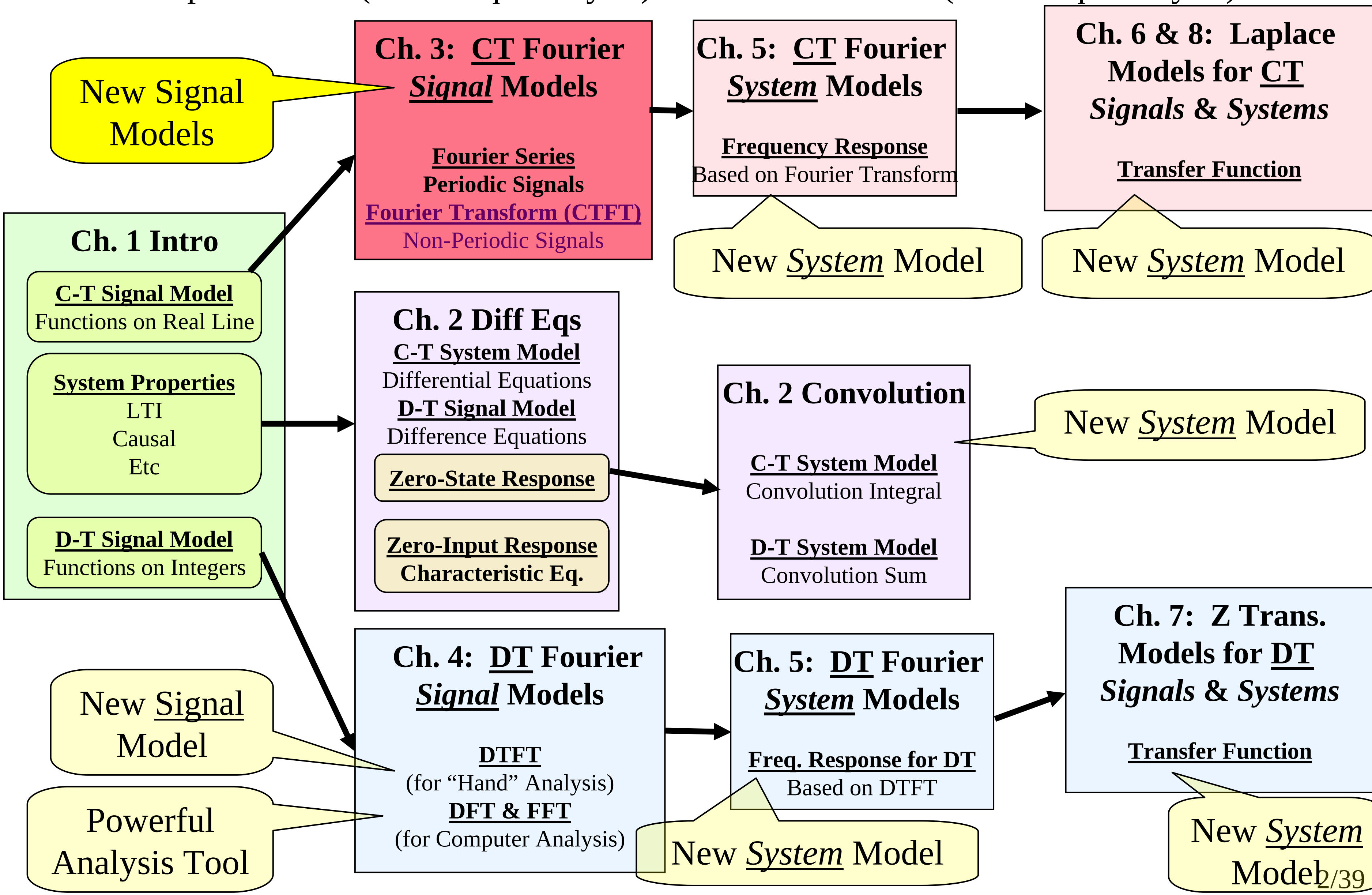
EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #15

- C-T Signals: Fourier Transform Properties
- Reading Assignment: Section 3.6 of Kamen and Heck

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



Fourier Transform Properties

Note: There are a few of these we won't cover….
see **Table on Website** or the inside front cover of the book for them.

I prefer that you use the tables on the website… they are better than the book's

As we have seen, finding the FT can be tedious (it can even be difficult)

But…there are certain properties that can often make things easier.

Also, these properties can sometimes be the key to understanding how the FT can be used in a given application.

So… even though these results may at first seem like "just boring math" they are important tools that let signal processing engineers understand how to build things like cell phones, radars, mp3 processing, etc.

1. **<u>Linearity</u>** **(Supremely Important)**

Gets used virtually all the time!!

If $x(t) \leftrightarrow X(\omega)$ & $y(t) \leftrightarrow Y(\omega)$

then $\left[ax(t) + by(t)\right] \leftrightarrow \left[aX(\omega) + bY(\omega)\right]$

Another way to write this property:

$$\mathfrak{F}\left\{ax(t) + by(t)\right\} = a\mathfrak{F}\left\{x(t)\right\} + b\mathfrak{F}\left\{y(t)\right\}$$

<u>To see why</u>: $\mathfrak{F}\left\{ax(t) + by(t)\right\} = \int_{-\infty}^{\infty} \left[ax(t) + by(t)\right] e^{-j\omega t} dt$

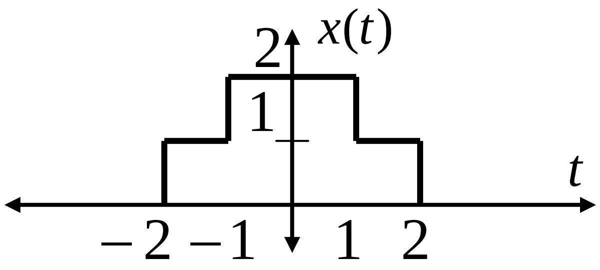
Use Defn of FT

$$= a\underbrace{\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}_{= X(\omega)} + b\underbrace{\int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt}_{= Y(\omega)}$$

By standard Property of Integral of sum of functions

By Defn of FT

Example Application of "Linearity of FT": Suppose we need to find the FT of the following signal…



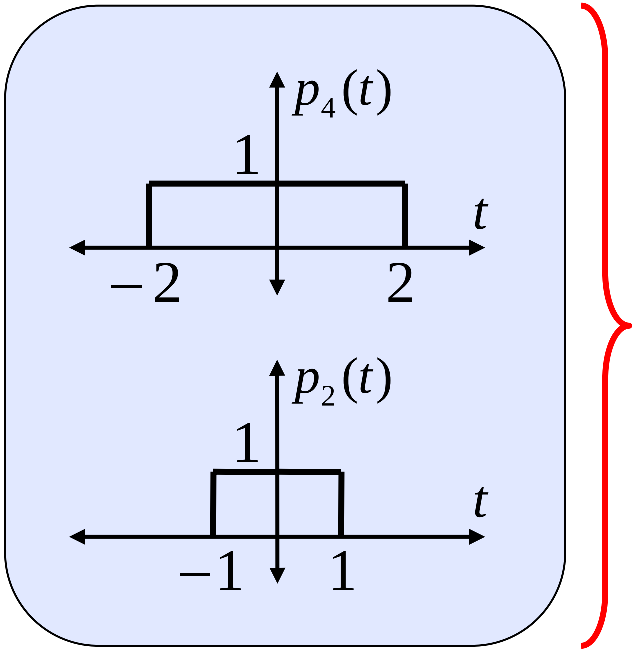
Finding this using straight-forward application of the definition of FT is not difficult but it is tedious:

$$\mathfrak{F}\{x(t)\} = \int_{-2}^{-1} e^{-j\omega t} dt + 2\int_{-1}^{1} e^{-j\omega t} dt + \int_{1}^{2} e^{-j\omega t} dt$$

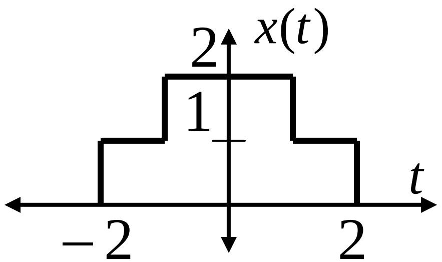
So… we look for short-cuts:

- One way is to recognize that each of these integrals is basically the same
- Another way is to break $x(t)$ down into a sum of signals on our table!!!

Break a complicated signal down into simple signals before finding FT:



Add
to get



Mathematically we write: $x(t) = p_4(t) + p_2(t)$ $\Rightarrow$ $X(\omega) = P_4(\omega) + P_2(\omega)$

From FT Table we have a <u>known</u> result for the FT of a pulse, so…

$$\boxed{X(\omega) = 4\,\mathrm{sinc}\left(\frac{2\omega}{\pi}\right) + 2\,\mathrm{sinc}\left(\frac{\omega}{\pi}\right)}$$

2. Time Shift (Really Important!)

Used often to understand practical issues that arise in audio, communications, radar, etc.

$$\boxed{\text{If} \quad x(t) \leftrightarrow X(\omega) \quad \text{then} \quad x(t-c) \leftrightarrow X(\omega)e^{-jc\omega}}$$

Note: If $c > 0$ then $x(t - c)$ is a **delay** of $x(t)$

So… what does this *mean*??

First… it does nothing to the magnitude of the FT: $\left|X(\omega)e^{-j\omega c}\right| = \left|X(\omega)\right|$

That means that a shift doesn't change "how much" we need of each of the sinusoids we build with

Second… it does change the phase of the FT: $\angle\{X(\omega)e^{-jc\omega}\} = \angle X(\omega) + \angle e^{-jc\omega}$

$$= \angle X(\omega) + \underbrace{c\omega}$$

Line of slope –c

Phase shift increases linearly as the frequency increases

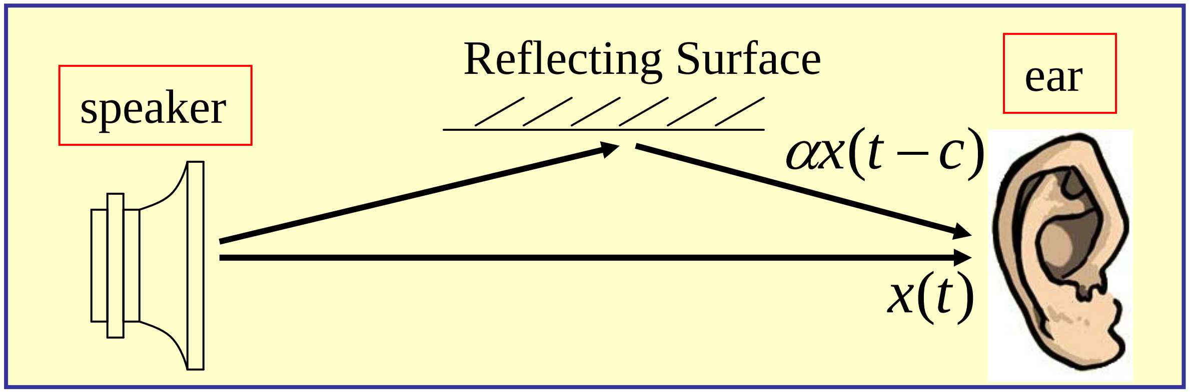
This gets added to original phase

Shift of Time Signal ⇔ "Linear" Phase Shift of Frequency Components

Example Application of Time Shift Property: Room acoustics.

Practical Questions: Why do some rooms sound bad? Why can you fix this by using a “graphic equalizer” to “boost” some frequencies and “cut” others?

Very simple case of a single reflection:



$c > 0$ Delayed signal

$0 < \alpha < 1$ Attenuated signal

So… You hear: $y(t) = x(t) + \alpha x(t-c)$ instead of just $x(t)$

Use linearity and time shift to get the FT at your ear:

$$Y(\omega) = \mathfrak{F}\{x(t) + \alpha x(t-c)\} = \mathfrak{F}\{x(t)\} + \alpha \mathfrak{F}\{x(t-c)\}$$

$$= X(\omega) + \alpha X(\omega) e^{-j\omega c}$$

$$\boxed{Y(\omega) = X(\omega)\left[1 + \alpha e^{-j\omega c}\right]}$$

This is the FT of what you hear…

It gives an equation that shows how the reflection affects what you hear!!!!

The big picture!

$$|Y(\omega)| = |X(\omega)|\underbrace{\left|1+\alpha e^{-j\omega c}\right|}_{\equiv |H(\omega)|}$$

$|H(\omega)|$ changes shape of $|X(\omega)|$

The room changes how much of each frequency you hear…

Let's look closer at |*H*(ω)| to see what it does…

Using Euler's formula gives Rectangular Form

$$|H(\omega)| = \left|1+\alpha e^{-jc\omega}\right| = \left|1+\alpha\cos(c\omega) - j\alpha\sin(c\omega)\right|$$

$$= \sqrt{(1+\alpha\cos(\omega c))^2 + \alpha^2\sin^2(\omega c)} = \sqrt{1+2\alpha\cos(\omega c) + \underbrace{\alpha^2\cos^2(\omega c)+\alpha^2\sin^2(\omega c)}_{=\alpha^2}}$$

$mag = \sqrt{(\mathrm{Re})^2 + (\mathrm{Im})^2}$

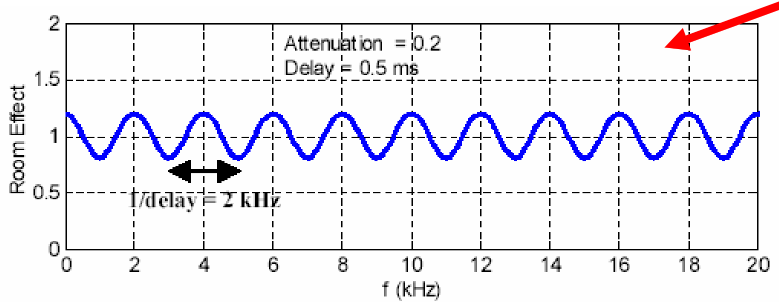
Expand 1st squared term

Use Trig ID

$$\boxed{|H(\omega)| = \sqrt{(1+\alpha^2) + 2\alpha\cos(\omega c)}}$$

The big picture… revisited:

$$\boxed{|Y(\omega)| = |X(\omega)|\sqrt{(1+\alpha^2) + 2\alpha\cos(\omega c)}}$$



Effect of the room… what does it look like as a function of frequency?? The cosine term makes it wiggle up and down… and the value of c controls how fast it wiggles up and down

$\boxed{\text{Spacing} = 1/c \text{ Hz}}$

"Dip-to-Dip"

"Peak-to-Peak"

c controls spacing between dips/peaks

α controls depth/height of dips/peaks

The next 3 slides explore these effects

<u>What is a typical value for delay c???</u>

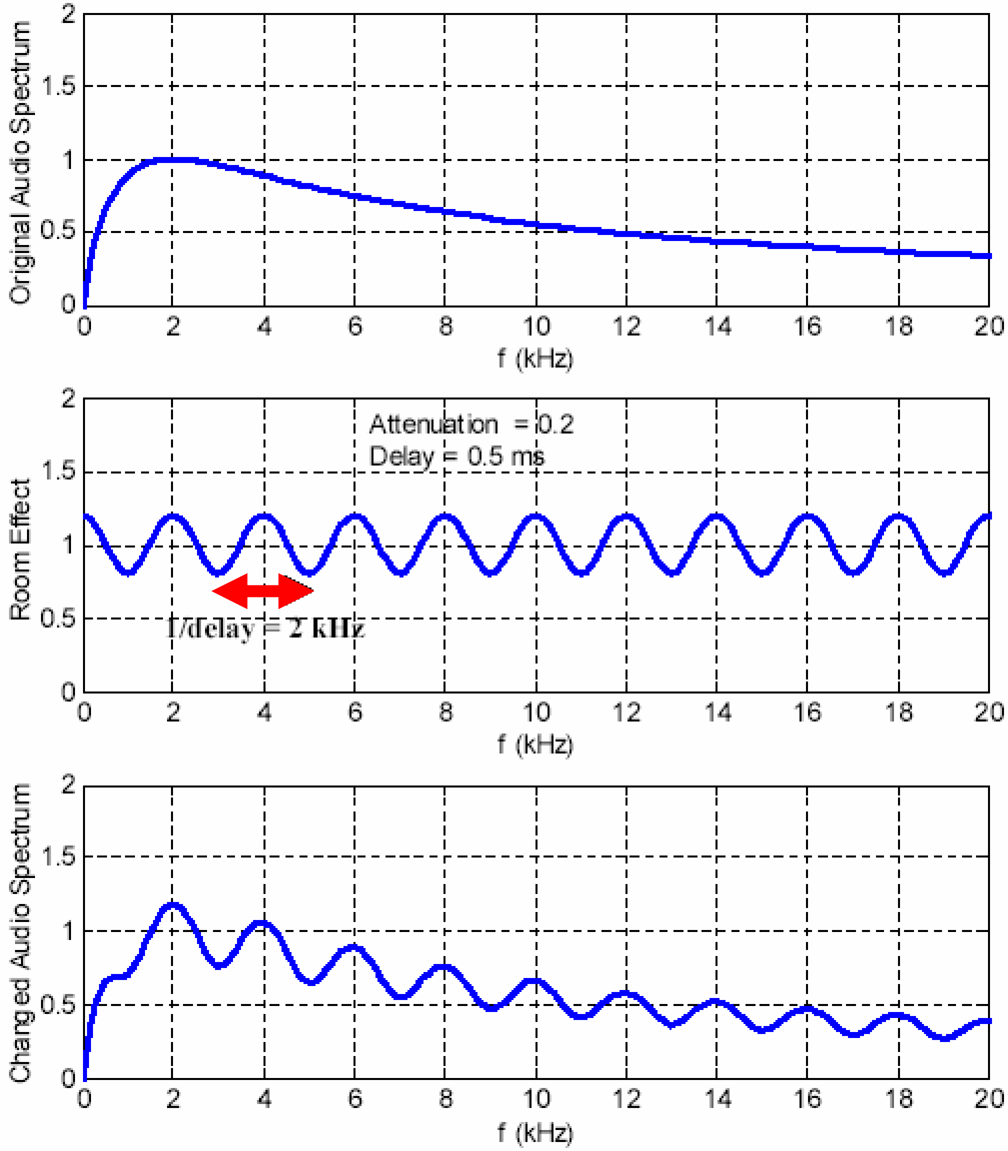
Speed of sound in air ≈ 340 m/s

Typical difference in distance ≈ 0.167m

$$c = \frac{0.167\text{m}}{340\text{m/s}} = 0.5m\text{sec}$$

➔ Spacing = 2 kHz

Attenuation: $\alpha = 0.2$ Delay: $c = 0.5$ ms (Spacing = 1/0.5e-3 = 2 kHz)



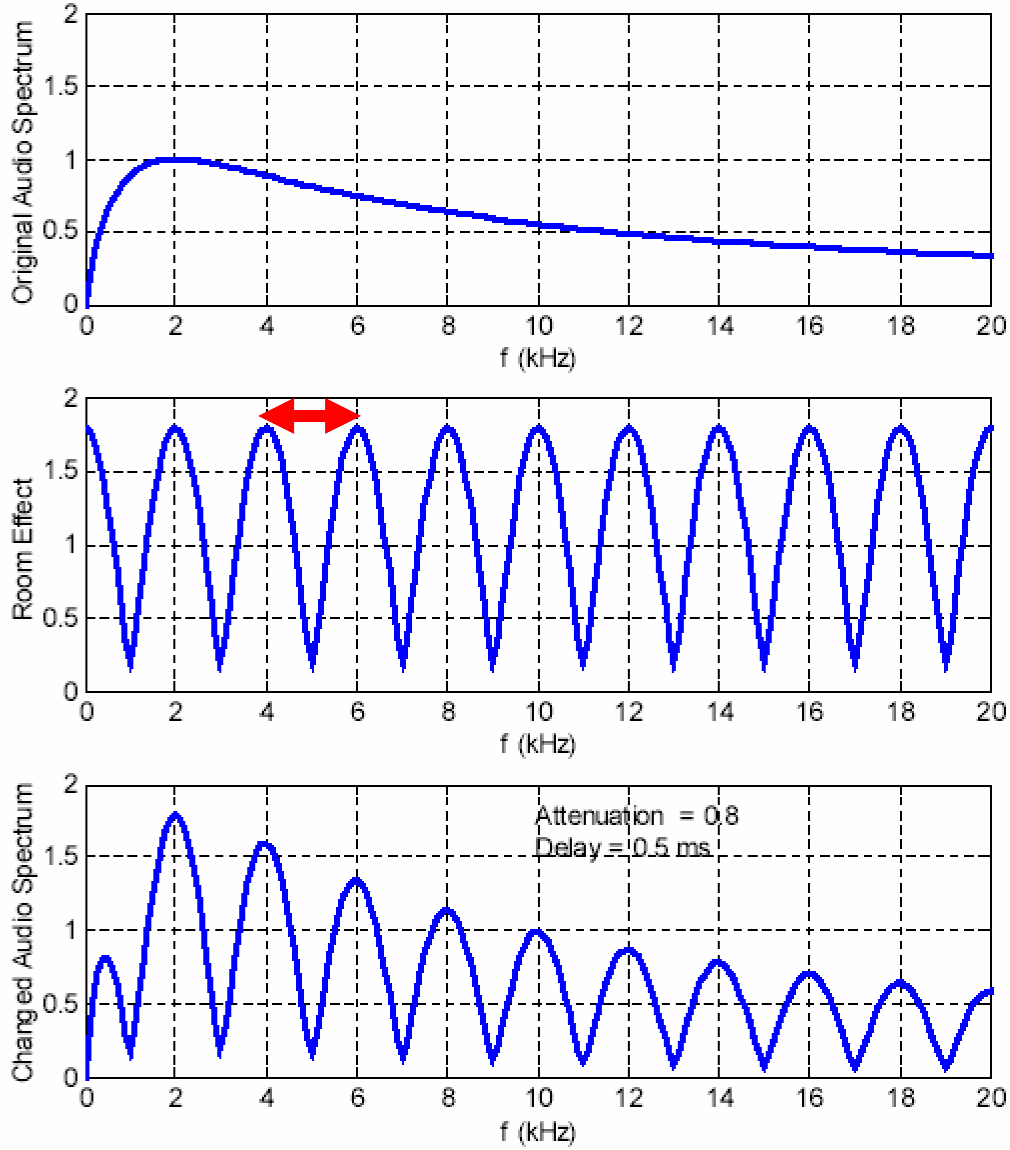
FT magnitude at the speaker (a made-up spectrum… but kind of like audio)

$|H(\omega)|$… the effect of the room

FT magnitude at your ear… room gives slight boosts and cuts at closely spaced locations

Longer delay causes closer spacing… so more dips/peaks over audio range!

Attenuation: $\alpha = 0.8$ Delay: c = 0.5 ms (Spacing = 1/0.5e-3 = 2 kHz)



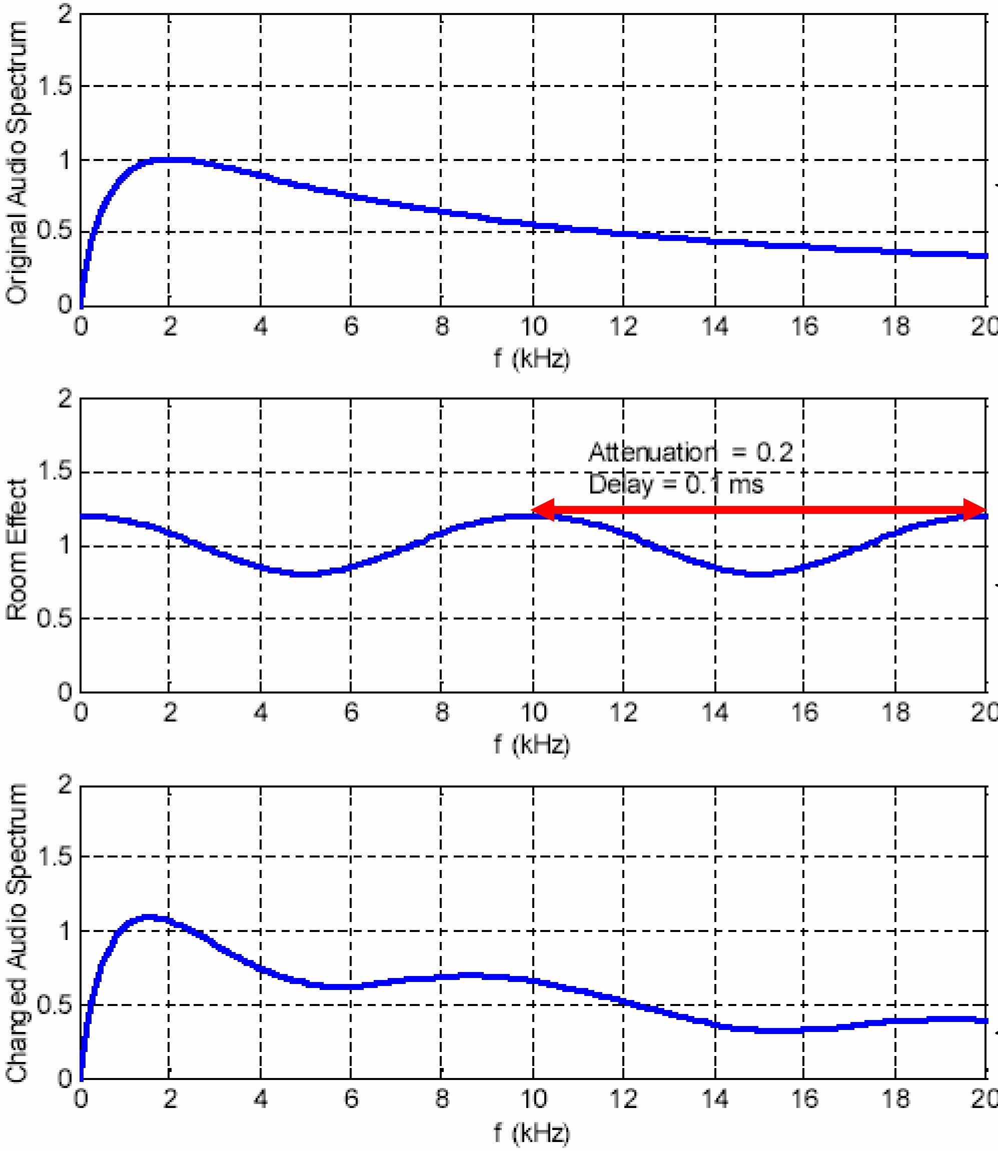
FT magnitude at the speaker

$|H(\omega)|$… the effect of the room

FT magnitude at your ear… room gives large boosts and cuts at closely spaced locations

Stronger reflection causes bigger boosts/cuts!!

Attenuation: $\alpha = 0.2$ Delay: $c = 0.1$ ms (Spacing = 1/0.1e-3 = 10 kHz)



FT magnitude at the speaker

$|H(\omega)|$… the effect of the room

FT magnitude at your ear… room gives small boosts and cuts at widely spaced locations

Shorter delay causes wider spacing… so fewer dips/peaks over audio range!

Matlab Code to create the previous plots

```
function room_delay(atten,delay)

f=0:100:20000;  %    Freq range: 0 Hz to 20 kHz
w=2*pi*f;       %    convert to rad/sec

H=abs(1 + atten*exp(-j*w*delay));  % Compute Room Effect

% Make up a fictitious audio spectrum
X=50000*w./((2*pi*2000+w)).^2;

% Now do plots
subplot(3,1,1)   % splits figure into 3 subplots, pick 1st one
plot(f/1000,X)   % note f converted into k Hz
xlabel('f (kHz)')
ylabel('Original Audio Spectrum')
axis([0 20 0 2])  % set axis ranges as desired
grid   % put grid lines on

subplot(3,1,2)   % splits figure into 3 subplots, pick 2nd one
plot(f/1000,H)
xlabel('f (kHz)')
ylabel('Room Effect')
axis([0 20 0 2])
grid

subplot(3,1,3)   % splits figure into 3 subplots, pick 3rd one
plot(f/1000,H.*X)
xlabel('f (kHz)')
ylabel('Changed Audio Spectrum')
axis([0 20 0 2])
grid
```

3. Time Scaling (Important)

Q: If $x(t) \leftrightarrow X(\omega)$, then $x(at) \leftrightarrow ???$ for $a \neq 0$

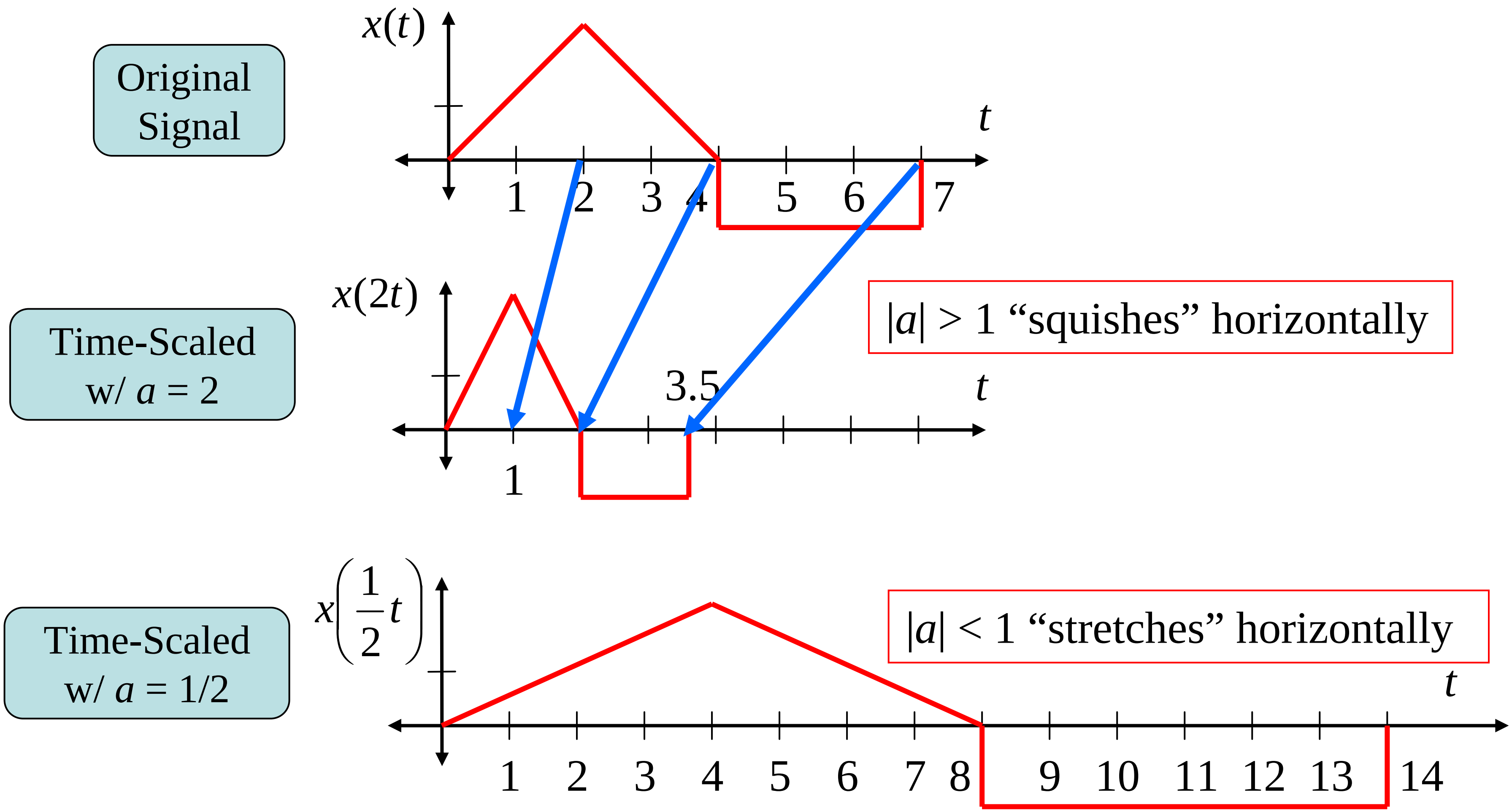
A: $\boxed{x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)}$

If the time signal is Time Scaled by *a*

Then… The FT is Freq. Scaled by 1/a

An interesting "duality"!!!

To explore this FT property…first, what does $x(at)$ look like?



$|a| > 1$ makes it "wiggle" faster $\Rightarrow$ need <u>more</u> high frequencies

$|a| < 1$ makes it "wiggle" slower $\Rightarrow$ need <u>less</u> high frequencies

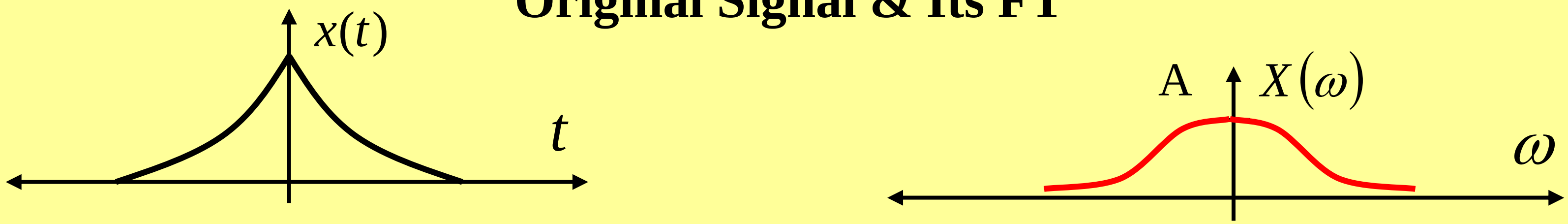
<u>When |*a*| > 1 $\Rightarrow$ |1/*a*| < 1</u>

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Time Signal is Squished

FT is Stretched Horizontally and Reduced Vertically

Original Signal & Its FT



Squished Signal & Its FT

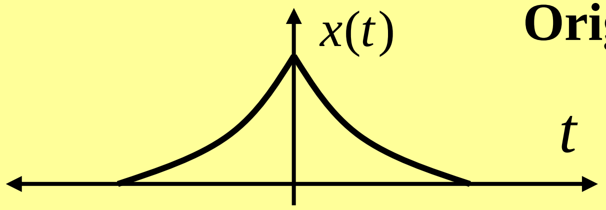
<u>When $|a| < 1 \;\Rightarrow\; |1/a| > 1$</u>

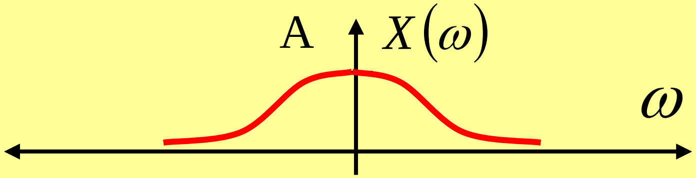
$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Time Signal is Stretched

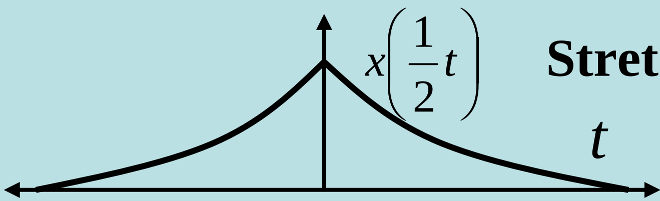
FT is Squished Horizontally
and Increased Vertically

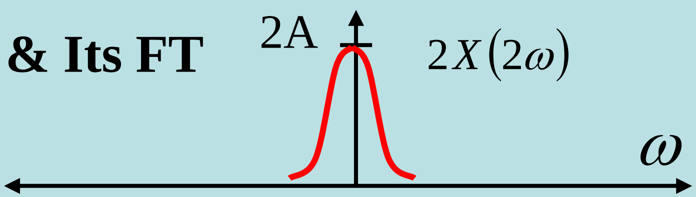
Original Signal & Its FT





Stretched Signal & Its FT





<u>Rough</u> Rule of Thumb we can extract from this property:

$\uparrow$ Duration $\Rightarrow$ $\downarrow$ Bandwidth

$\downarrow$ Duration $\Rightarrow$ $\uparrow$ Bandwidth

Very Short Signals *tend* to take up Wide Bandwidth

<u>4. Time Reversal</u> (Special case of time scaling: $a = -1$)

$$\boxed{x(-t) \leftrightarrow X(-\omega)}$$

double conjugate

<u>Note:</u> $X(-\omega) = \int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt \;\; = \overline{\overline{\int_{-\infty}^{\infty} x(t)e^{+j\omega t}dt}}$ = "No Change"

$$= \overline{\int_{-\infty}^{\infty} \overline{x(t)}\,\overline{e^{+j\omega t}}dt}$$

Conjugate changes to $-j$

$= x(t)$ if $x(t)$ is real

$$= \overline{\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt} = \overline{X(\omega)}$$

<u>Recall</u>: conjugation doesn't change abs. value but negates the angle

So if $x(t)$ is <u>real</u>, then we get the <u>special case</u>:

$$\boxed{x(-t) \leftrightarrow \overline{X(\omega)}}$$

$$\boxed{\begin{gathered} \left|\overline{X(\omega)}\right| = \left|X(\omega)\right| \\ \angle\overline{X(\omega)} = -\angle X(\omega) \end{gathered}}$$

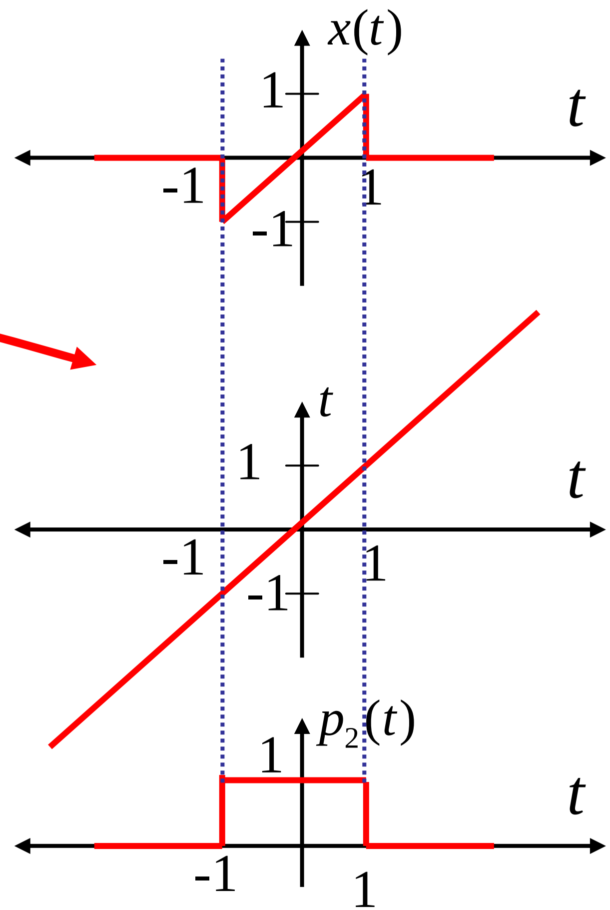
5. Multiply signal by t^n

$$\boxed{t^n x(t) \leftrightarrow (j)^n \frac{d^n X(\omega)}{d\omega^n}} \qquad n = \text{positive integer}$$

This property is mostly useful for finding the FT of typical signals.

Example Find $X(\omega)$ for this $x(t)$

Notice that: $x(t) = t p_2(t)$



So… we can use this property as follows:

From entry on FT Table with $\tau = 2$.

$$X(\omega) = j\frac{d}{d\omega}P_2(\omega) = j\frac{d}{d\omega}\left(2\,\text{sinc}\left(\frac{\omega}{\pi}\right)\right)$$

$$= j2\left[\frac{\omega\cos(\omega) - \sin(\omega)}{\omega^2}\right]$$

Now… how do you get the derivative of the sinc???

Use the definition of sinc and then use the rule for the derivative of a quotient you learned in Calc I:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{df(x)}{dx} - f(x)\dfrac{dg(x)}{dx}}{g^2(x)}$$

6. Modulation Property

Super important!!!

Essential for understanding practical issues that arise in communications, radar, etc.

There are two forms of the modulation property…

1. **Complex Exponential Modulation** … simpler mathematics, doesn't *directly* describe real-world cases
2. **Real Sinusoid Modulation**… mathematics a bit more complicated, directly describes real-world cases

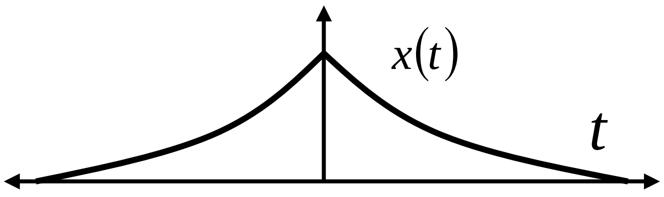
Euler's formula connects the two… so you often can use the Complex Exponential form to analyze real-world cases

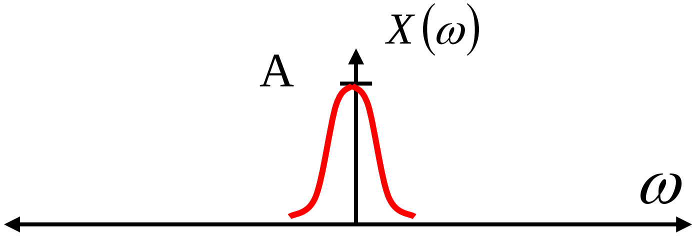
Complex Exponential Modulation Property

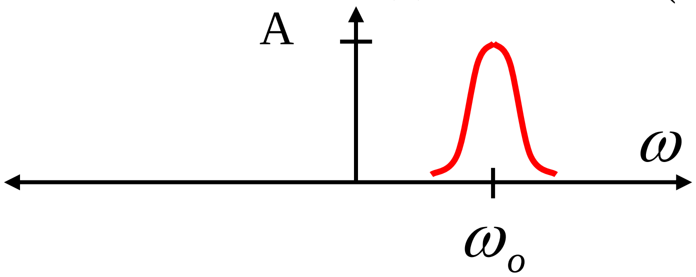
$$\boxed{x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)}$$

Multiply signal by a complex sinusoid

Shift the FT in frequency







Real Sinusoid Modulation

Based on Euler, Linearity property, & the Complex Exp. Modulation Property

$$\mathfrak{F}\{x(t)\cos(\omega_0 t)\} = \mathfrak{F}\left\{\frac{1}{2}\left[x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}\right]\right\}$$

Euler's Formula

$$= \frac{1}{2}\left[\mathfrak{F}\left\{\left[x(t)e^{j\omega_0 t}\right]\right\} + \mathfrak{F}\left\{\left[x(t)e^{-j\omega_0 t}\right]\right\}\right]$$

Linearity of FT

$$= \frac{1}{2}\left[X(\omega - \omega_o) + X(\omega + \omega_o)\right]$$

Comp. Exp. Mod.

The Result: $\boxed{x(t)\cos(\omega_0 t) \quad \leftrightarrow \quad \frac{1}{2}\left[X(\omega + \omega_0) + X(\omega - \omega_0)\right]}$

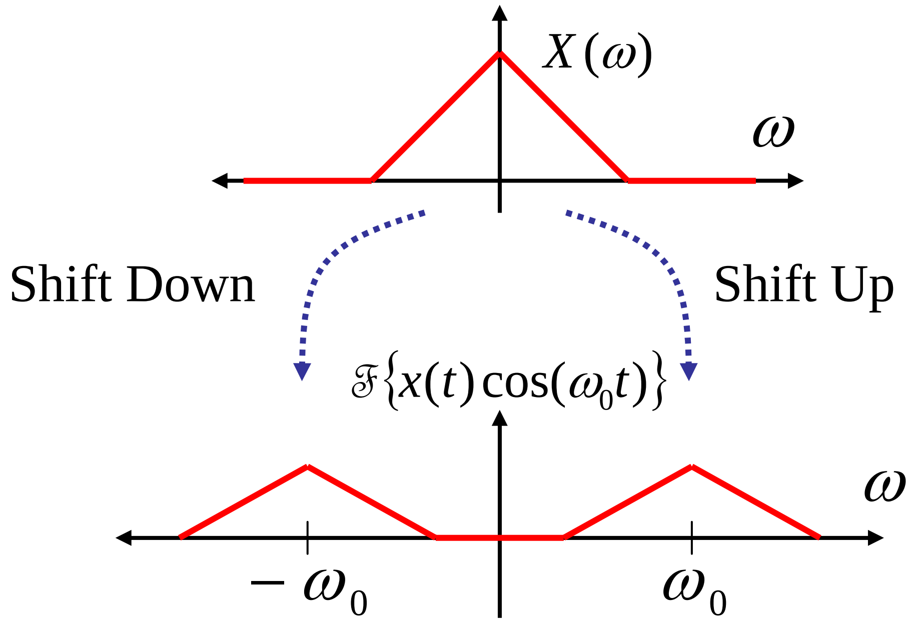
Shift Down Shift Up

Related Result: $\boxed{x(t)\sin(\omega_0 t) \quad \leftrightarrow \quad \frac{j}{2}\left[X(\omega + \omega_0) - X(\omega - \omega_0)\right]}$

Exercise: $x(t)\cos(\omega_0 t + \phi_0) \leftrightarrow ??$

Visualizing the Result

$$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[\underbrace{X(\omega-\omega_0)}_{\text{Shift up}} + \underbrace{X(\omega+\omega_0)}_{\text{Shift down}}\right]$$

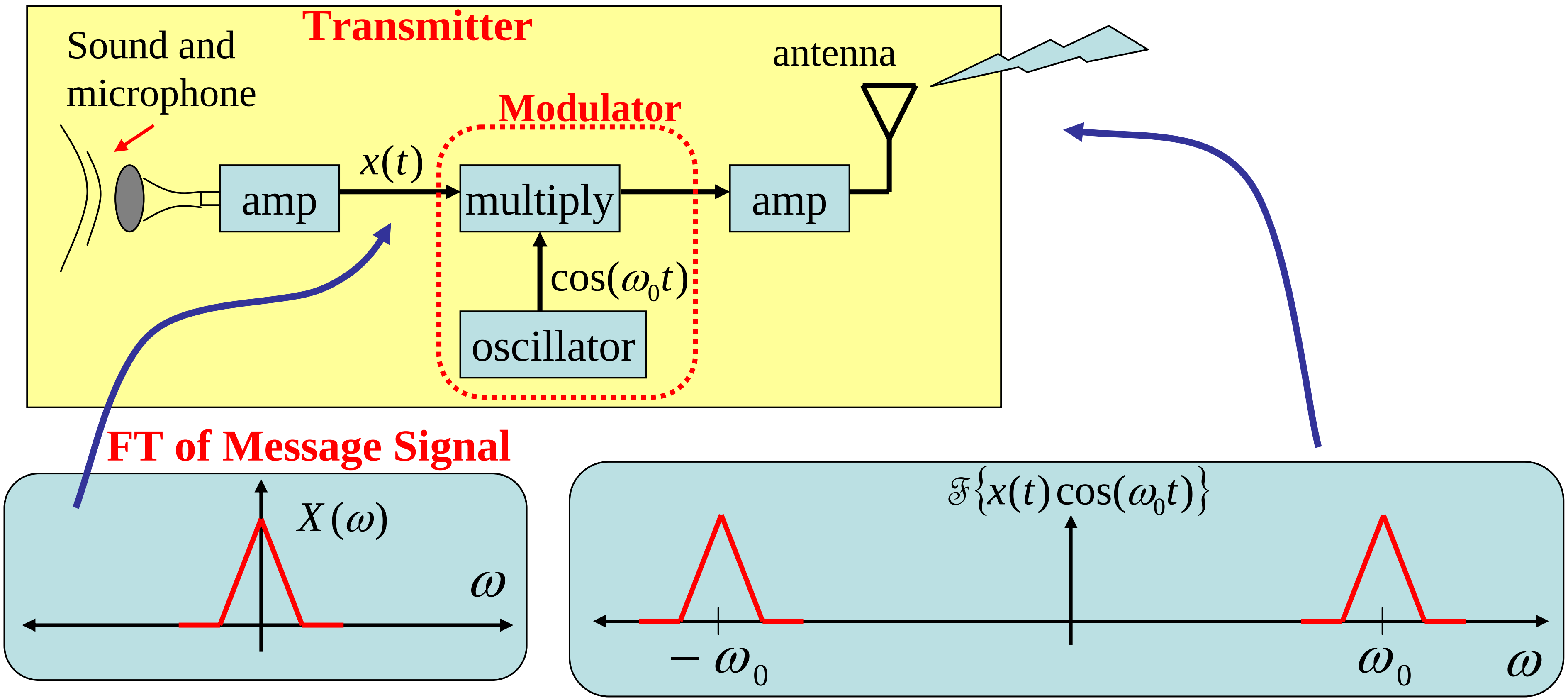


Interesting… This tells us how to move a signal's spectrum up to higher frequencies without changing the shape of the spectrum!!!

What is that good for??? Well… only high frequencies will radiate from an antenna and propagate as electromagnetic waves and then induce a signal in a receiving antenna….

<u>Application of Modulation Property to Radio Communication</u>

FT theory tells us what we need to do to make a **<u>simple</u>** radio system… *<u>then</u>* electronics can be built to perform the operations that the FT theory calls for:

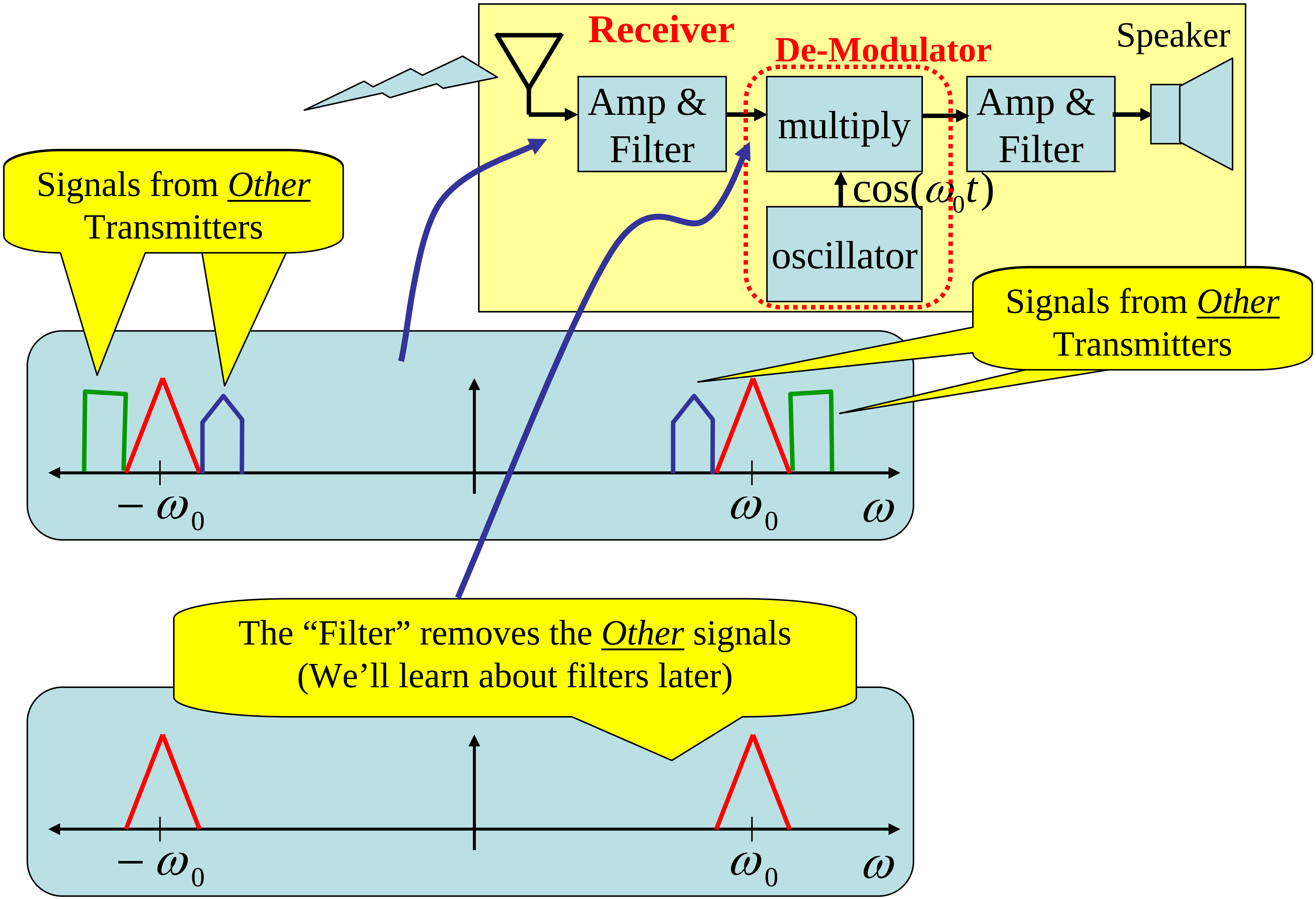


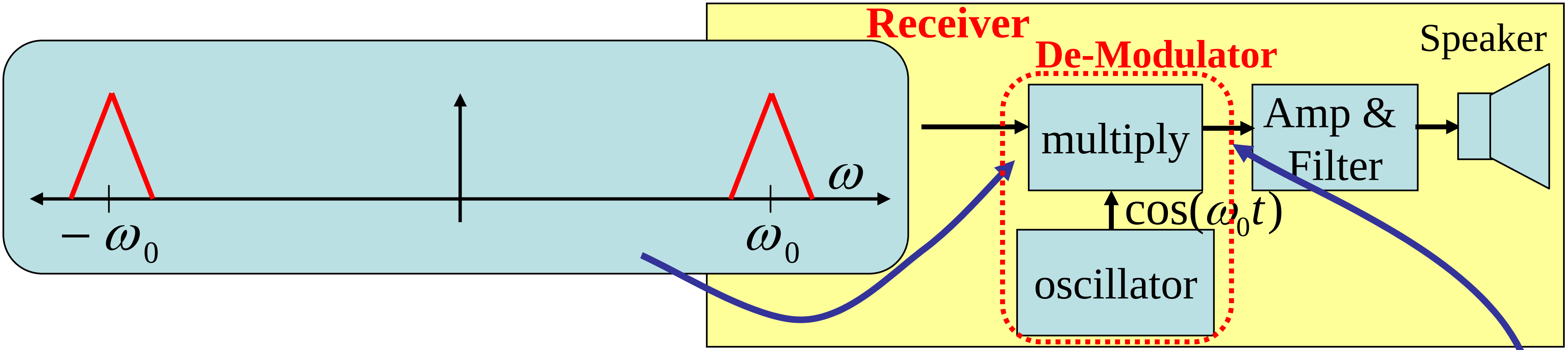
Choose f_0 > 10 kHz to enable efficient radiation (with $\omega_0 = 2\pi f_0$)

<u>AM Radio</u>: around 1 MHz <u>FM Radio</u>: around 100 MHz

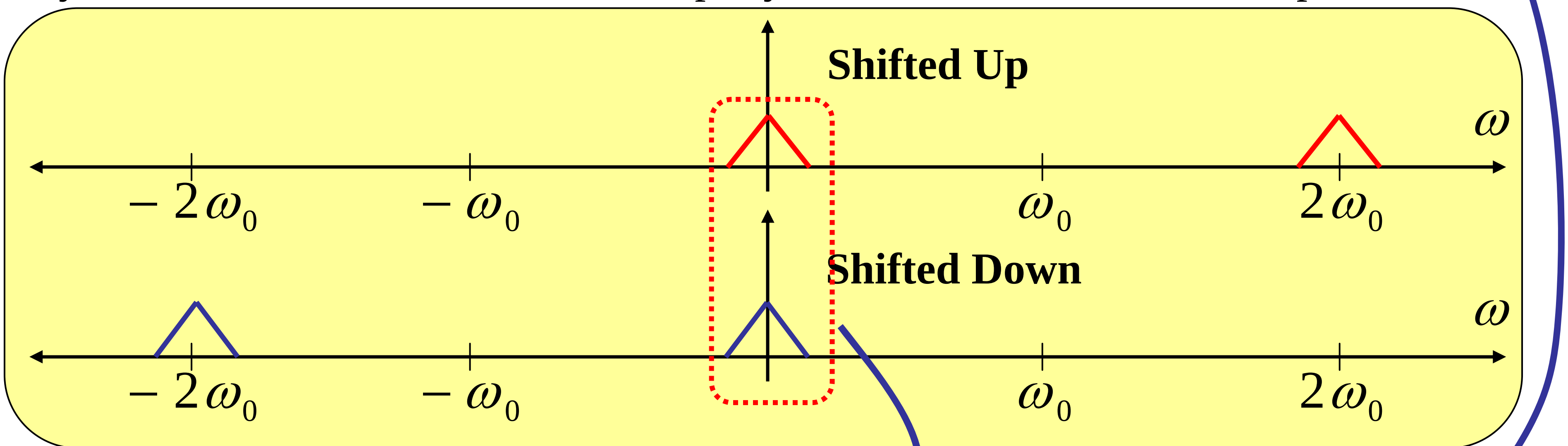
<u>Cell Phones</u>: around 900 MHz, around 1.8 GHz, around 1.9 GHz etc.

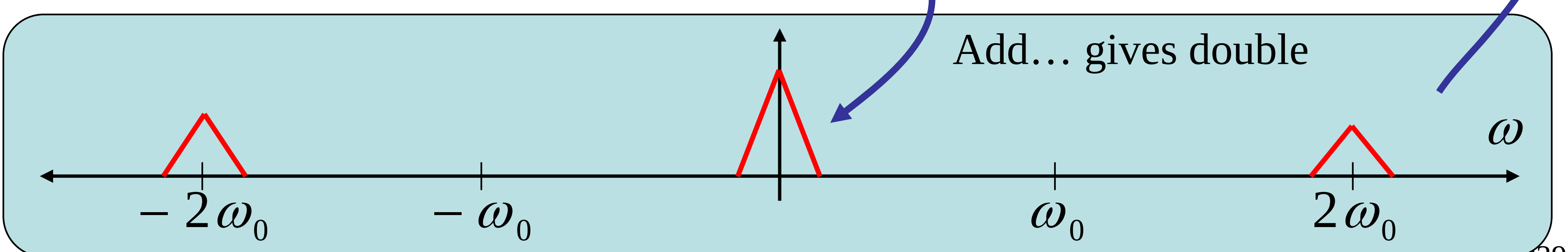
The next several slides show how these ideas are used to make a receiver:

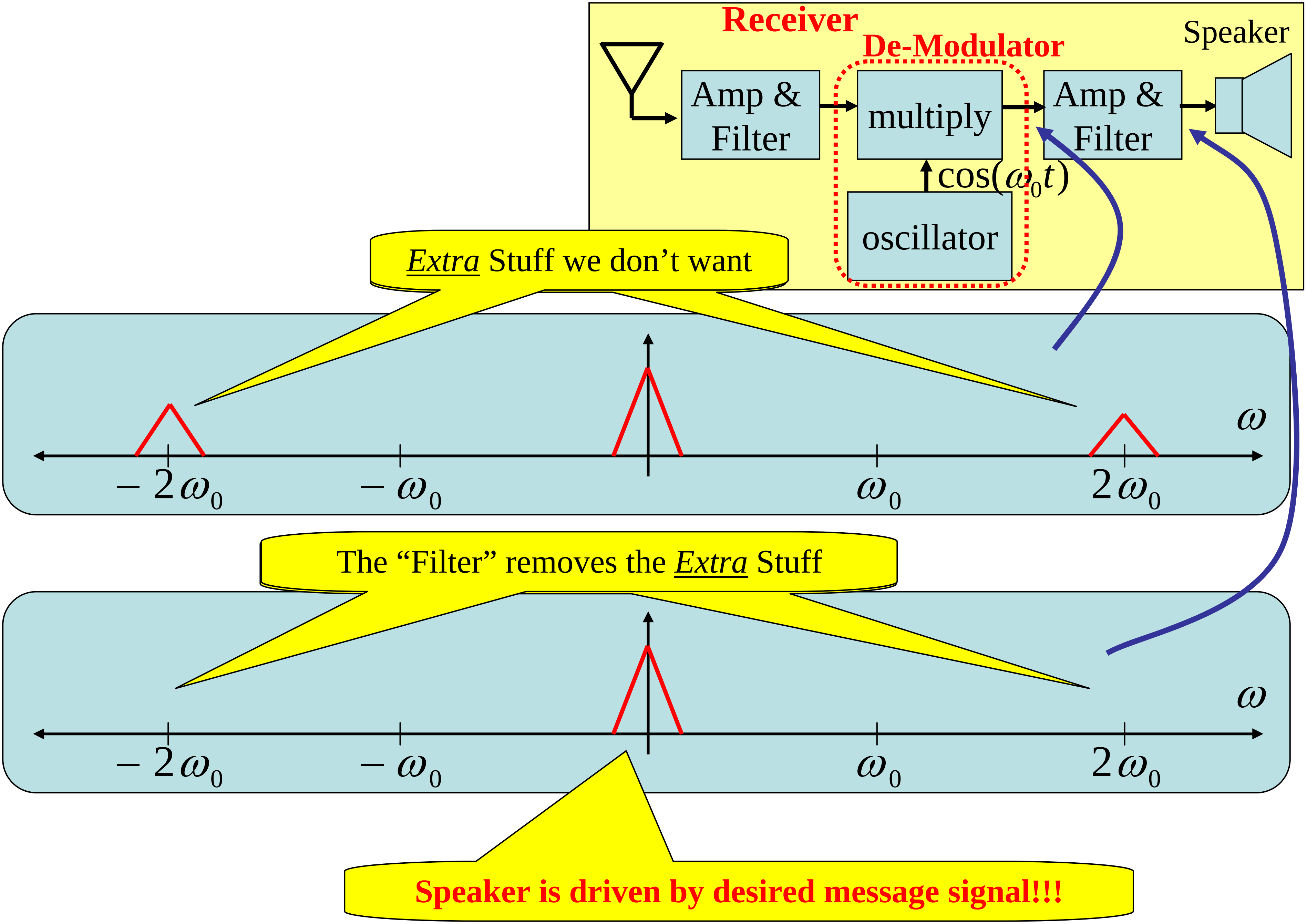




By the Real-Sinusoid Modulation Property… the De-Modulator shifts up & down:






Receiver
De-Modulator
Speaker
Amp & Filter
multiply
Amp & Filter
$\cos(\omega_0 t)$
oscillator
Extra Stuff we don't want
ω
$-2\omega_0$
$-\omega_0$
ω_0
$2\omega_0$
The "Filter" removes the *Extra* Stuff
ω
$-2\omega_0$
$-\omega_0$
ω_0
$2\omega_0$
Speaker is driven by desired message signal!!!

So… what have we seen in this example:

Using the Modulation property of the FT we saw…

1. Key Operation at Transmitter is up-shifting the message spectrum:
    a) FT Modulation Property tells the theory then we can build…
    b) "modulator" = oscillator and a multiplier circuit
2. Key Operation at Transmitter is down-shifting the received spectrum
    a) FT Modulation Property tells the theory then we can build…
    b) "de-modulator" = oscillator and a multiplier circuit
    c) But… the FT modulation property theory also shows that we need filters to get rid of "extra spectrum" stuff
        i. So… one thing we still need to figure out is how to deal with these filters…
        ii. Filters are a specific "system" and we still have a lot to learn about Systems…
        iii. That is the subject of much of the rest of this course!!!

<u>7. Convolution Property</u> (The Most Important FT Property!!!)

The ramifications of this property are the subject of the <u>entire</u> Ch. 5 and continues into <u>all</u> the other chapters!!!

It is this property that makes us study the FT!!

Mathematically we state this property like this:

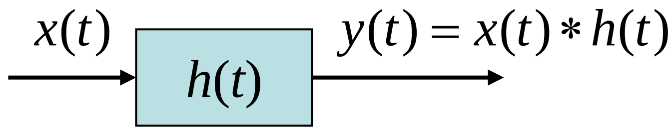
$$\boxed{x(t) * h(t) \quad \leftrightarrow \quad X(\omega)H(\omega)}$$

Another way of stating this is:

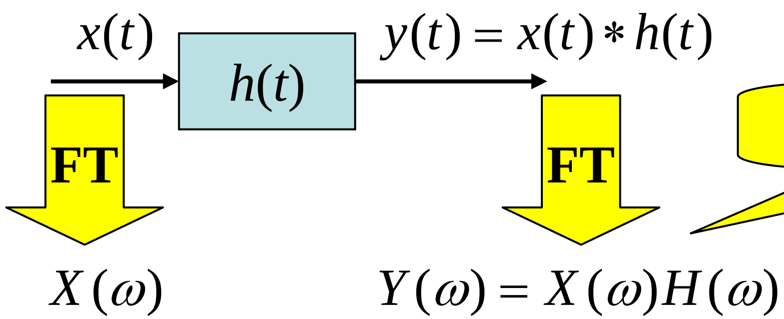
$$\boxed{\mathscr{F}\left\{x(t) * h(t)\right\} = X(\omega)H(\omega)}$$

Now… what does this mean and why is it so important??!!

Recall that convolution is used to described what comes out of an LTI system:



Now we can take the FT of the input and the output to see how we can view the system behavior “in the frequency domain”:

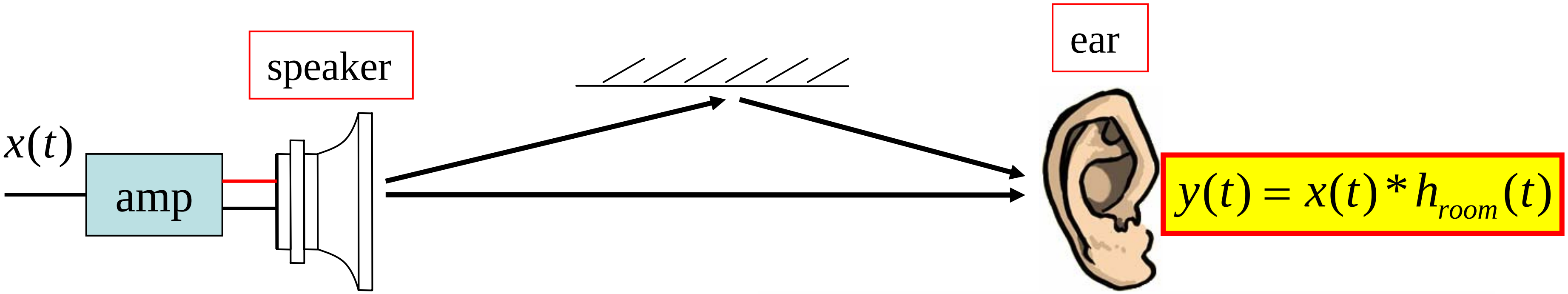


Use the Conv. Property!!

System’s $H(\omega)$ changes the shape of the input’s $X(\omega)$ via <u>multiplication</u> to create output’s $Y(\omega)$

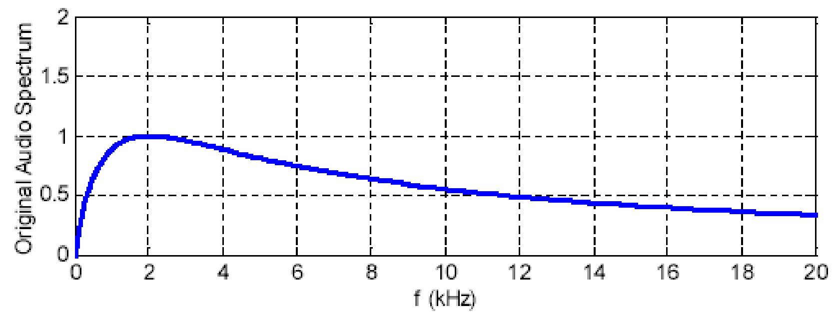
It is easier to think about and analyze the operation of a system using this “frequency domain” view because visualizing multiplication is easier than visualizing convolution

Let's revisit our "Room Acoustics" example:

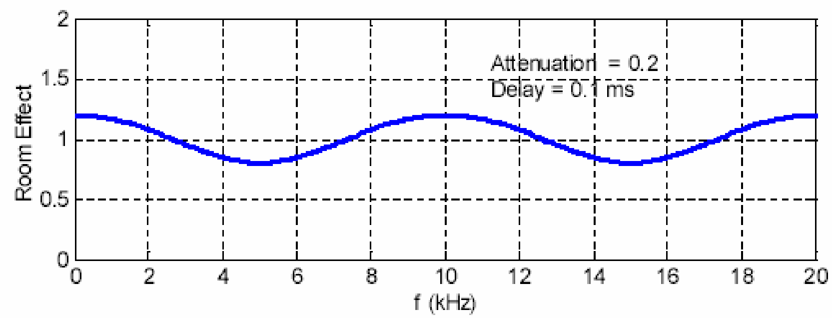


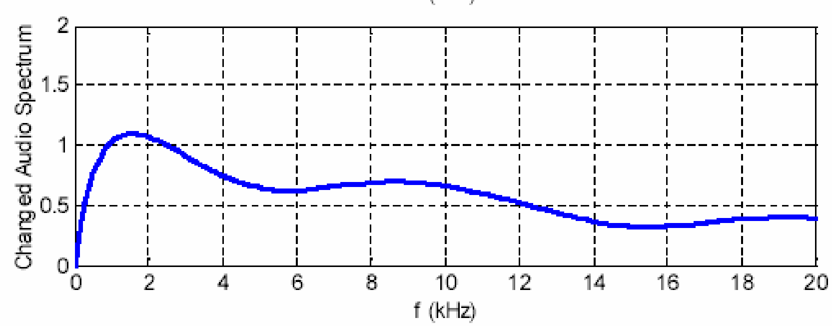
Recall:

$$|Y(\omega)| = |X(\omega)|\underbrace{|1+\alpha e^{-j\omega c}|}_{H_{\text{room}}(\omega)}$$



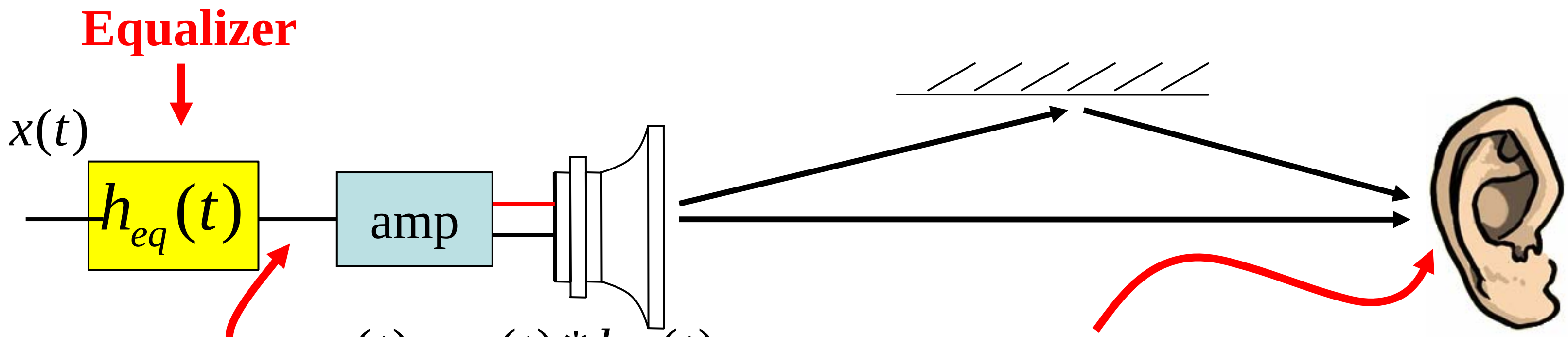
Plot of $|H_{\text{room}}(\omega)|$





What we hear is not right!!!

So, we fix it by putting in an "equalizer" (a system that fixes things)



$$x_2(t) = x(t) * h_{eq}(t)$$

$$\boxed{X_2(\omega) = H_{eq}(\omega) X(\omega)}$$

(by convolution property)

$$y(t) = x_2(t) * h_{room}(t)$$

$$= \left[x(t) * h_{eq}(t)\right] * h_{room}(t)$$

$$\boxed{\begin{aligned} Y(\omega) &= H_{room}(\omega) X_2(\omega) \\ &= H_{room}(\omega) H_{eq}(\omega) X(\omega) \end{aligned}}$$

(by convolution property, applied twice!)

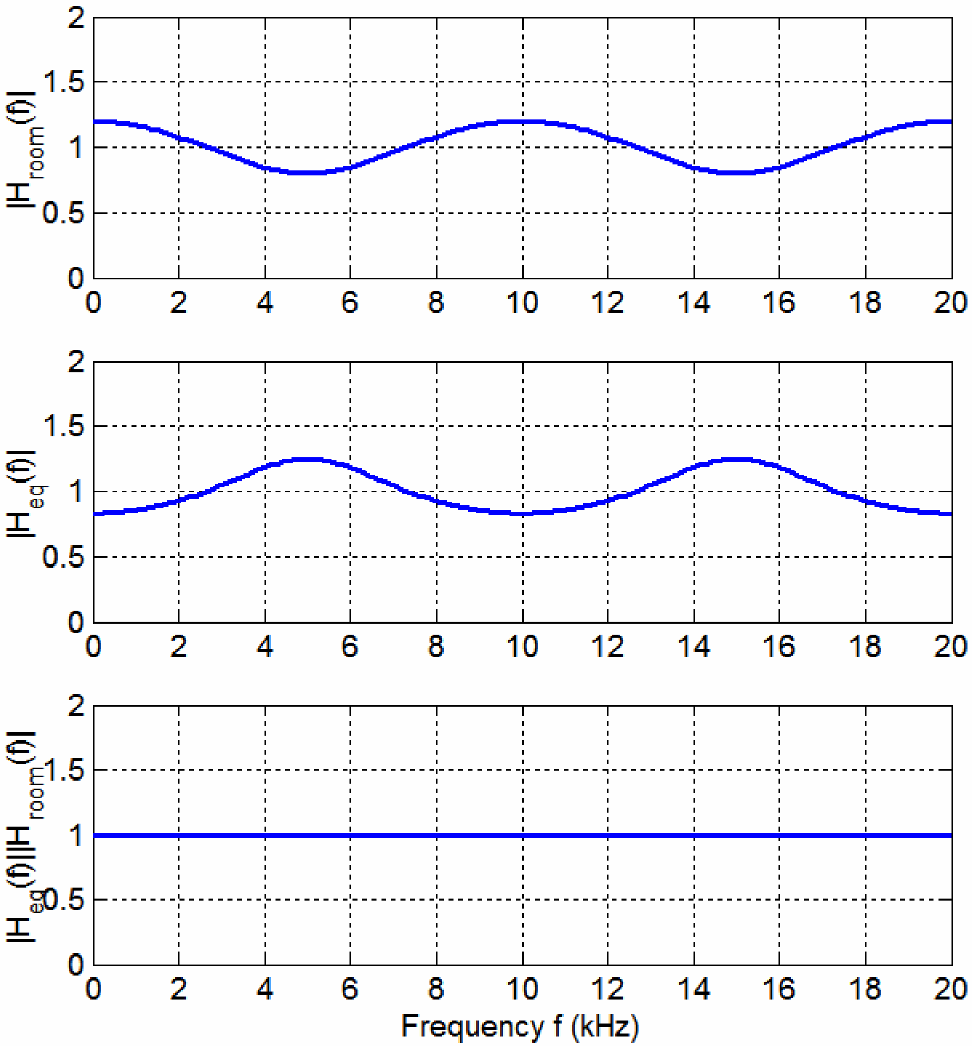
Then: $$\boxed{|Y(\omega)| = |X(\omega)| \underbrace{|H_{eq}(\omega)| \underbrace{|1 + \alpha e^{-j\omega c}|}_{\text{Recall: Peaks and dips}}}_{\text{Want this whole thing to be = 1 so } |Y(\omega)| = |X(\omega)|}}$$

Equalizer's $|H_{eq}(\omega)|$ should peak at frequencies where the room's $|H_{room}(\omega)|$ dips and vice versa

Room

Equalizer

Room & Equalizer



8. Multiplication of Signals

$$x(t)y(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * Y(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)Y(\omega-\lambda)d\lambda$$

This is the "dual" of the convolution property!!!

"Convolution in the Time-Domain"	**gives**	"Multiplication in the Frequency-Domain"

"Multiplication in the Time-Domain"	**gives**	"Convolution in the Frequency-Domain"

9. Parseval's Theorem (Recall Parseval's Theorem for FS!)

$$\boxed{\underbrace{\int_{-\infty}^{\infty} |x(t)|^2 \, dt}_{\text{Energy computed in time domain}} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega}_{\text{Energy computed in frequency domain}}}$$

$$|x(t)|^2 \, dt$$

= energy at time t

$$|X(\omega)|^2 \frac{d\omega}{2\pi}$$

= energy at freq. ω

Generalized Parseval's Theorem:

$$\boxed{\int_{-\infty}^{\infty} x(t)\overline{y(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\overline{Y(\omega)}d\omega}$$

10. Duality:

$$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$x(t) \qquad\qquad X(\omega)$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

Both FT & IFT are pretty much the "<u>same</u> machine": $c\int_{-\infty}^{\infty} f(\lambda)e^{\pm j\lambda\xi}\,d\lambda$

So if there is a "time-to-frequency" property we would expect a virtually similar "frequency-to-time" property

<u>Illustration:</u> Delay Property: $x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$

Modulation Property: $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega-\omega_0)$

<u>Other Dual Properties:</u> (Multiply by t^n) vs. (Diff. in time domain)

(Convolution) vs. (Mult. of signals)

Also, this duality structure gives FT pairs that show duality.

Suppose we have a FT table that a FT Pair A… we can get the dual Pair B using the <u>general</u> Duality Property:

1. Take the FT side of (known) Pair A and replace ω by *t* and move it to the time-domain side of the table of the (unknown) Pair B.
2. Take the time-domain side of the (known) Pair A and replace *t* by –ω, multiply by 2π, and then move it to the FT side of the table of the (unknown) Pair B.

Here is an example… We found the FT pair for the pulse signal:

Pair A $\boxed{p_\tau(t)} \longleftrightarrow \boxed{\tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)}$

Step 1 **Step 2**

Pair B $\boxed{\tau \operatorname{sinc}\left(\frac{\tau t}{2\pi}\right)} \longleftrightarrow \boxed{2\pi p_\tau(\omega)}$

Here we have used the fact that $p_\tau(-\omega) = p_\tau(\omega)$