BINGHAMTON
UNIVERSITY
State University of New York

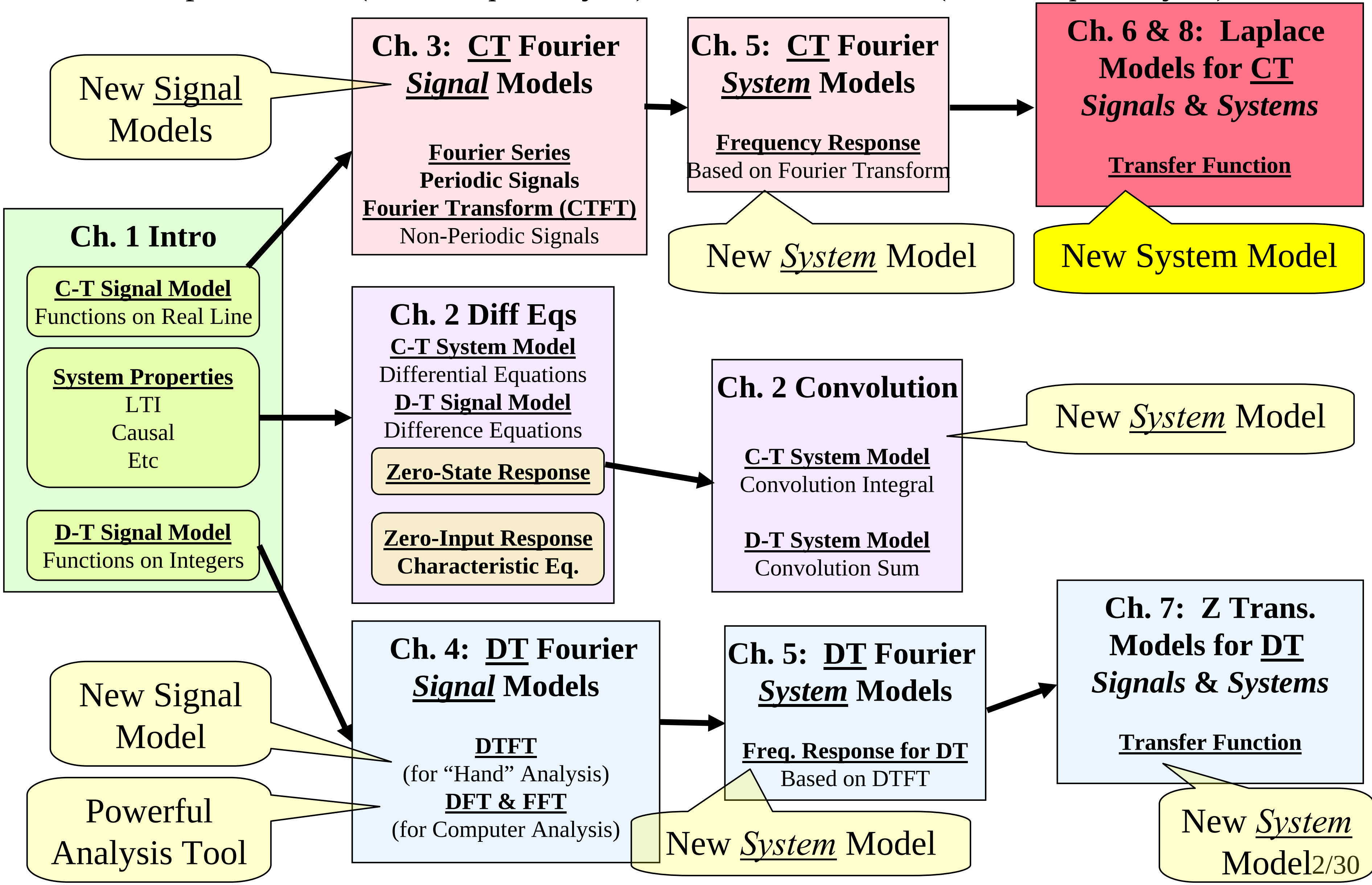
EECE 301
Signals & Systems
Prof. Mark Fowler

<u>Note Set #30</u>

- C-T Systems: Laplace Transform… and System Stability
- Reading Assignment: Section 8.1 of Kamen and Heck

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



Ch. 8: System analysis and design using the transfer function

We have seen that the system transfer function $H(s)$ plays an important role in the analysis of a system's output for a given input.

e.g. for the zero-state case:

$$Y(s) = X(s)H(s) \rightarrow y(t) = \mathcal{L}^{-1}\{X(s)H(s)\}$$

Much insight can be gained by looking at $H(s)$ and understanding how its structure will affect the form of $y(t)$.

Section 8.1: First we'll look at how $H(s)$ can tell us about a system's "stability"

Section 8.4: Then we'll see how the form of $H(s)$ can tell us about how the system output should behave

Section 8.5: Then we'll see how to design an $H(s)$ to give the desired frequency response.

Section 8.1 System Stability

Roughly speaking: A "stable" system is one whose output does not keep getting bigger and bigger in response to an input that does not keep getting bigger.

We can state this mathematically and then use our math models (e.g. $h(t)$ or $H(s)$) to determine if a system will be stable.

Math definition of stability

"Bounded-Input, Bounded-Output" (BIBO) stability:

A system is said to be BIBO stable if

for any bounded input: $|x(t)| \le C_1 < \infty$ $\quad \forall\, t \ge 0$ ($x(t)=0,\, t<0$)

the output remains bounded: $|y(t)| \le C_2 < \infty$ $\quad$ For SOME C_1 & C_2

Mathematical checks for stability:

The following are given without proof. They are all equivalent checks so you only need to test for one of them.

1. $\int_0^\infty |h(t)|\,dt < \infty$ "Absolutely Integrable"

2. All poles are in the "open left-half of the s-plane"

3. Routh-Hurwitz test (section 8.2, we'll skip it)

Discussion of Stability Checks

$$\int_0^\infty |h(t)| dt < \infty$$ can only happen if $|h(t)|$ decays fast enough.

This is hard to check!!!

Consider a system with $h(t) = \sin(\omega_0 t) u(t)$

This is not absolutely integrable… as you integrate its absolute value from 0 to ∞ the value of the integral keeps growing without bound…

So the system having this impulse response is not BIBO stable… it is unstable… that means that there is a bounded input that will (eventually) drive the system's output to infinity.

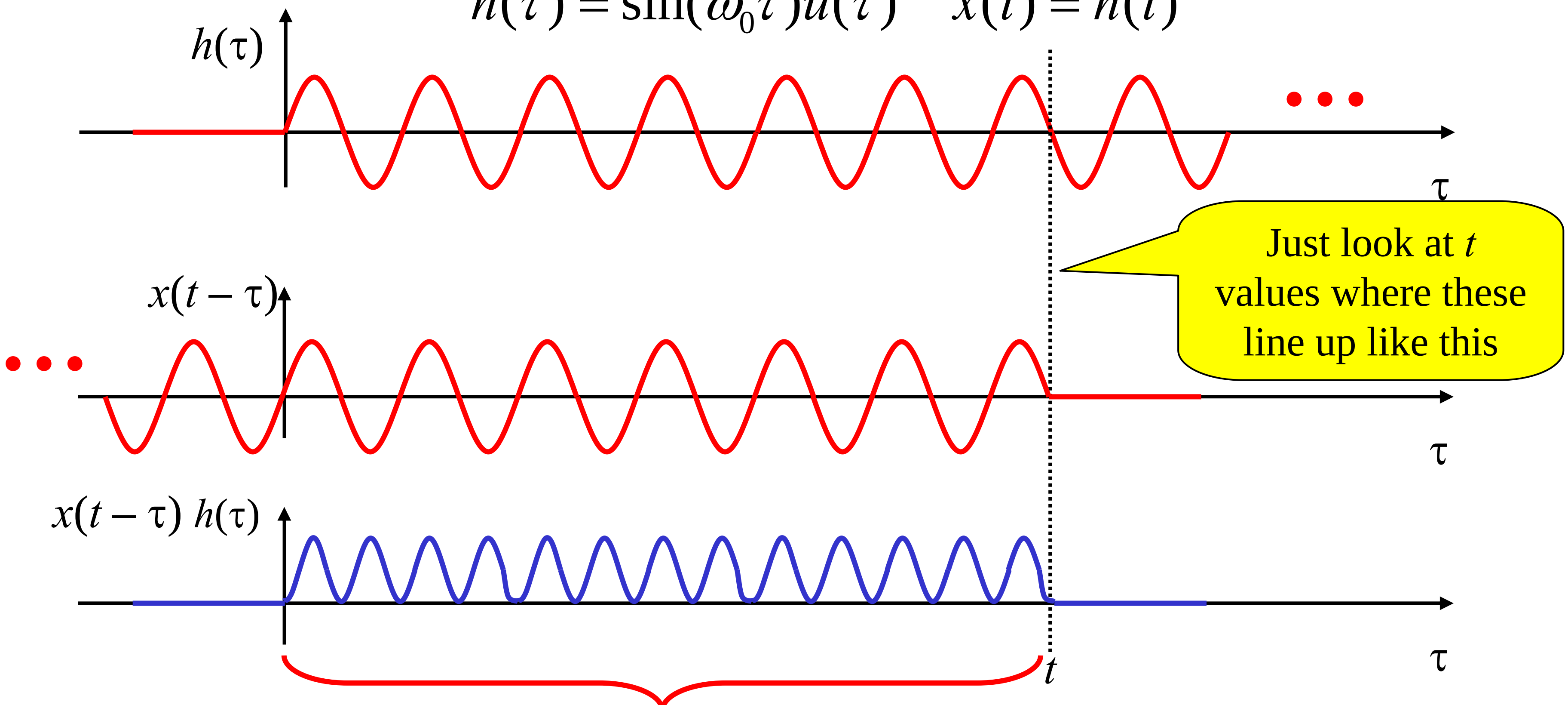
Note: Real H/W will "encounter problems" long before the output "goes to infinity"!!!

For this system, if you put in $x(t) = \sin(\omega_0 t) u(t)$

the system output is driven to infinity…

We can verify this claim using convolution:

$$h(\tau) = \sin(\omega_0 \tau)u(\tau) \quad x(t) = h(t)$$

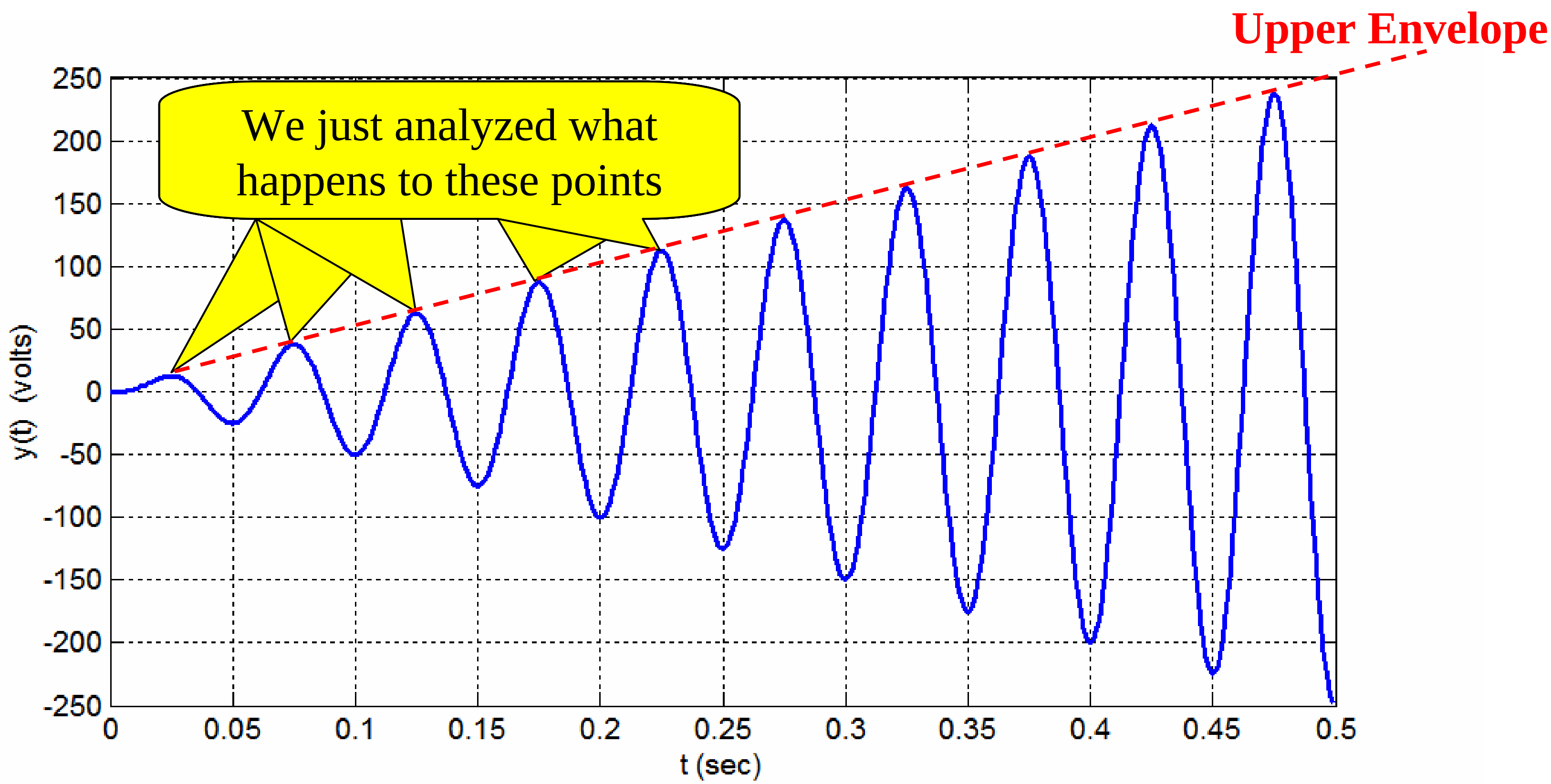


To get *y*(*t*)… Integrate… gives area of the humps… as *t* grows you get more and more humps… output grows without bound!!!

Thus, the upper "envelope" of the output grows by the area of one of those humps each time we increase *t* by one-half the period of the sinusoid…

So… the upper envelope grows linearly with time… it grows without bound!!!

Computing the complete convolution gives something like this:



Doing numerical calculations like this are helpful… but they don't PROVE that something happens… the plot by itself does not show what happens after 0.5 seconds!! As far as we know… anything could happen out there!!

That is the value of mathematical analysis (based on sound models, of course!)

So… we ***can*** use the absolute integrability check … but it is not that easy to do in general. It also doesn't tell us much about why a system is stable.

Well… if the impulse response can be used to check for stability… it should be no surprise that the transfer function can also be used!!!

Consider: $$H(s) = \frac{b_M s^M + b_{M-1}s^{M-1} + \ldots + b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \ldots + a_1 s + a_0} = \underbrace{H_1(s) + \ldots + H_p(s)}$$

After Partial Fraction Expansion

A term for each real pole
A term for each complex pole pair

After the Inverse LT we get:

$$h(t) = h_1(t) + \ldots + h_p(t)$$

From our understanding of partial fraction expansion we know what each of these terms can look like:

For Real Pole

$h_i(t) = c_i e^{p_i t} u(t)$ distinct pole

$h_i(t) = c_i t^k e^{p_i t} u(t)$ k - repeated poles

For Complex Pole Pair

$h_i(t) = c_i e^{\sigma_i t} \cos(\omega_i t + \theta_i) u(t)$ distinct pole

$h_i(t) = c_i t^k e^{p_i t} \cos(\omega_\mathrm{i} \mathrm{t} + \theta_\mathrm{i}) u(t)$ k - repeated poles

So what do these look like?

For the distinct pole cases:

$$h_i(t) = c_i e^{p_i t} \qquad h_i(t) = c_i e^{\sigma_i t} \cos(\omega_i t + \theta_i) u(t)$$

This decays if $p_i < 0$

This decays if $\sigma_i < 0$

We can capture the condition for decay with:

"the real-part of the pole is negative"

Although we won't prove it here… it can be shown that this decay is fast enough to ensure "absolute integrability"… and thus stability.

For the repeated pole cases:

$$h_i(t) = c_i t^k e^{p_i t} u(t) \qquad h_i(t) = c_i t^k e^{p_i t} \cos(\omega_i t + \theta_i) u(t)$$

We've got a race!!! The t^k terms are "going up" and the exponentials are "going down" for poles whose real parts are negative… WHO WINS???

The decaying exponential wins!!!

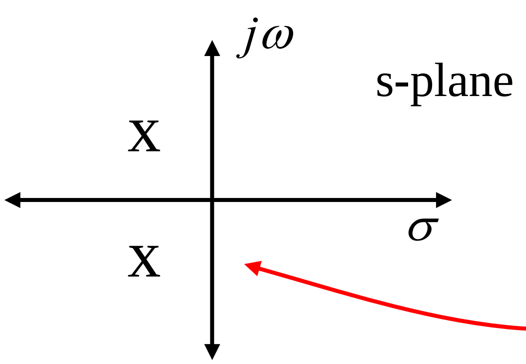
So we have argued that… a system is stable if all its poles have negative real parts

$$\mathrm{Re}\{p_i\} < 0,\ \forall i$$

Stability condition (not proved here!)

An N^{th} order system (with N poles $p_i = 1, 2, \ldots, N$) is stable if and only if

$$\boxed{\mathrm{Re}\{p_i\} < 0 \ \text{ for all } i = 1, 2, \ldots, N}$$



Can't be ***on*** the $j\omega$ axis

All poles must be in the "open" left-hand side of the s-plane

A system is said to be "marginally stable" if it has at least one distinct pole on the $j\omega$ axis but no repeated poles on $j\omega$

Marginally stable systems are not a good idea in most applications because they will still tend to get large outputs under certain conditions…

Comment: Just because the model satisfies BIBO stability, this does not mean the physical system will have no problems:

$$|y(t)| < C_2$$

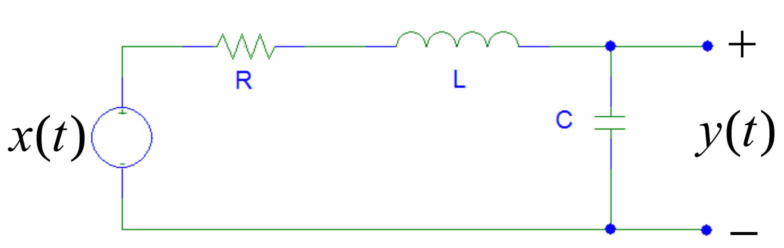
C_2 might exceed physical limits ⇒ system breaks

Summary

- In most applications we desire a stable system
- We can easily check for stability by looking to see where the system's poles are

Also… We won't show it here, but every stable system has an $H(s)$ whose ROC includes the $j\omega$ axis… therefore, the FT-based frequency response $H(\omega)$ exists.

Example: RLC circuit **Assess this circuit for stability**



Recall: $$H(s) = \frac{1/LC}{s^2 + (R/L)s + (1/LC)}$$

So all we need to do is find where its poles are!!!!

Use the Quadratic Formula on the denominator of $H(s)$:

$$p_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

From this result we can find out how the component values impact the stability of this system.

So we'll now systematically analyze this… we'll look at four cases.

<u>Case 1</u>:

$$p_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

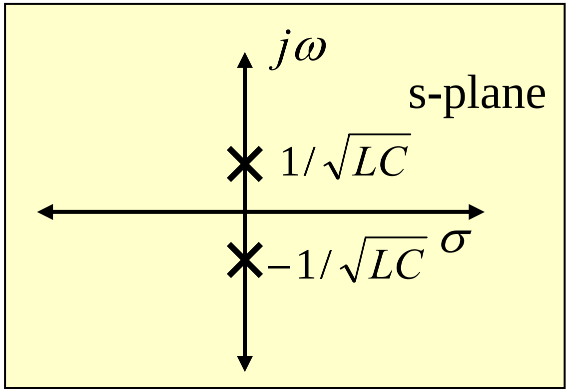
<u>Note:</u> If $R = 0$ $\Rightarrow p_{1,2} = \pm\sqrt{-\frac{1}{LC}} = \pm j\sqrt{\frac{1}{LC}}$

Causes Imaginary Part

So for this case… the poles are purely imaginary.

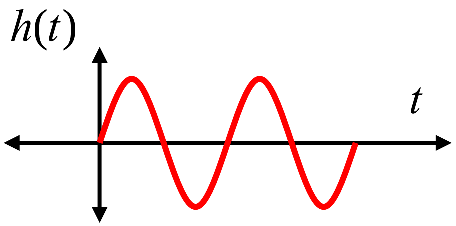
They lie on the $j\omega$ axis…

Marginally Stable



For this case we then have: $H(s) = \dfrac{1/LC}{s^2 + (1/LC)}$

From the LT table we get: $h(t) = \dfrac{1}{\sqrt{LC}} \sin\left[\left(1/\sqrt{LC}\right)t\right]$



Looks like an oscillator!!

But we can't <u>really</u> build an oscillator this way because all <u>real</u> *L* & *C* have "parasitic" resistance (need to use special electronic circuits to get an osc.)

All real passive circuits are stable

Case 2:

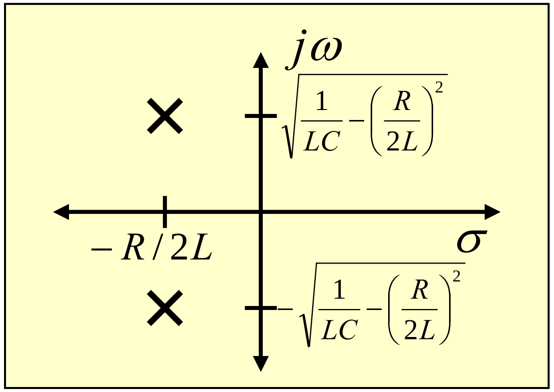
$$p_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Note: If $\left(\frac{\mathrm{R}}{2\mathrm{L}}\right)^2 < \frac{1}{LC}$ the $\sqrt{\sim\sim}$ term is < 0

$$\Rightarrow p_{1,2} = \frac{-R}{2L} \pm j\omega_0$$

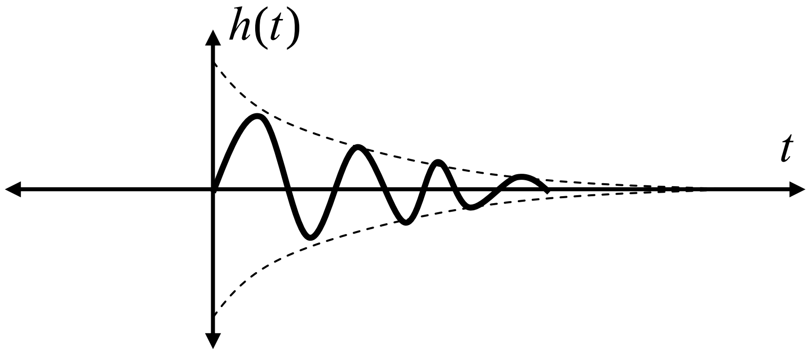
$$\omega_0 \triangleq \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$\mathrm{Re}\{p_i\} < 0 \Rightarrow stable$



For this case the inverse LT gives the following form:

$$h(t) = Ae^{(-R/2L)t}\cos(\omega_0 t + \theta)$$



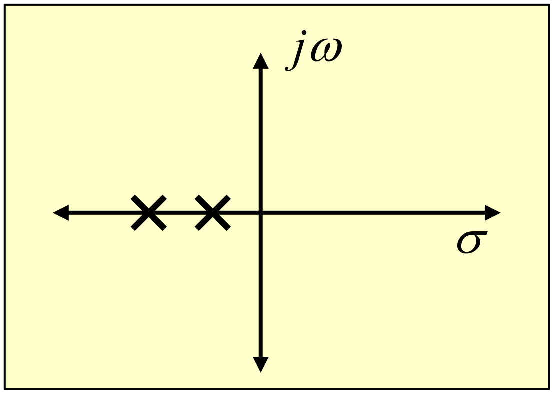
$$0 < R < \frac{2L}{\sqrt{LC}}$$

Stable with decaying oscillatory $h(t)$

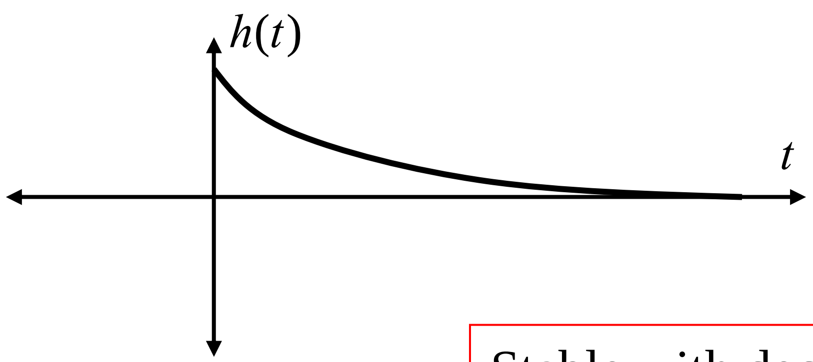
Case 3:

$$p_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Note: If $\left(\frac{\mathrm{R}}{2\mathrm{L}}\right)^2 > \frac{1}{LC}$, $p_{1,2}$ are real, distinct & < 0



$$h(t) = c_1 e^{\sigma_1 t} + c_2 e^{\sigma_2 t}$$



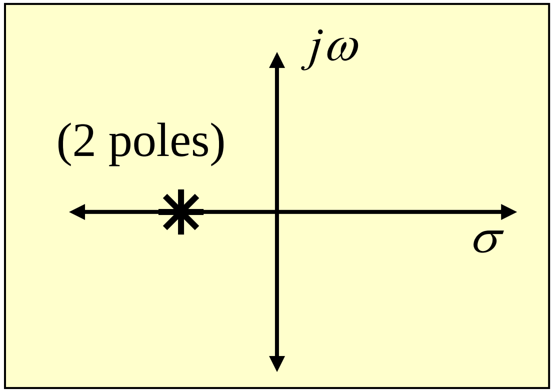
$$R > \frac{2L}{\sqrt{LC}}$$

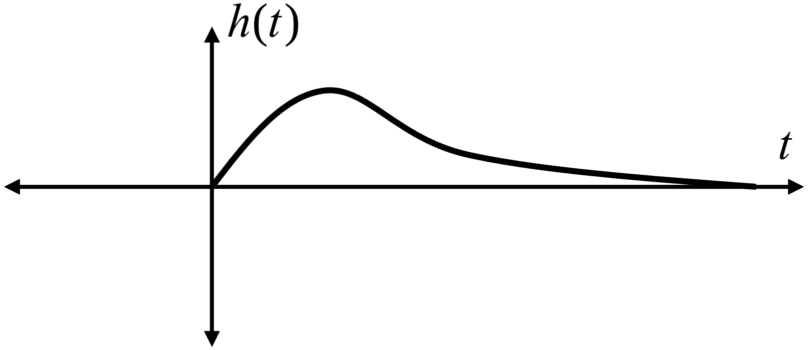
Stable with decaying non-oscillatory $h(t)$

Case 4:

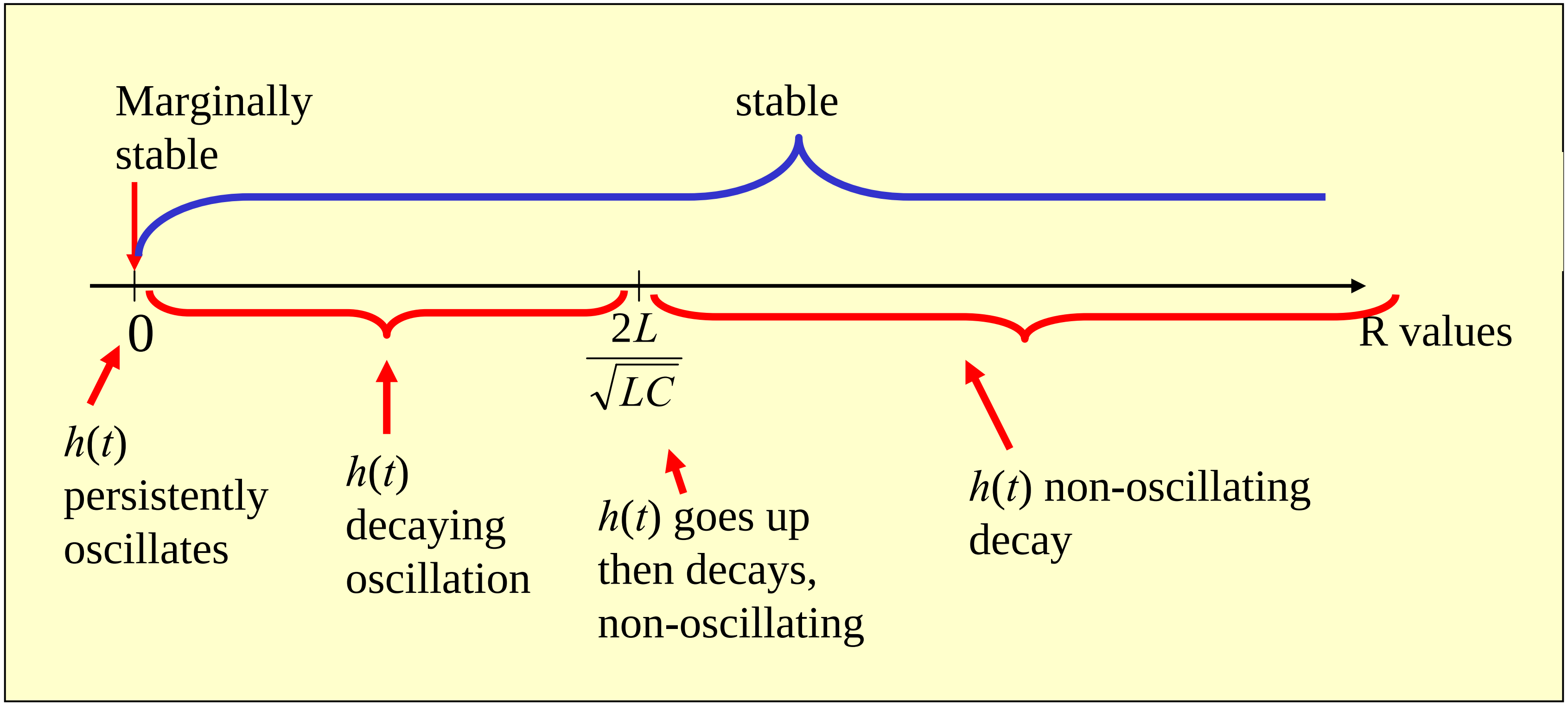
$$p_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\text{Note: If} \quad R = \frac{2L}{\sqrt{LC}}, \quad p_1 = p_2 \ \text{repeated poles} \ \& < 0$$





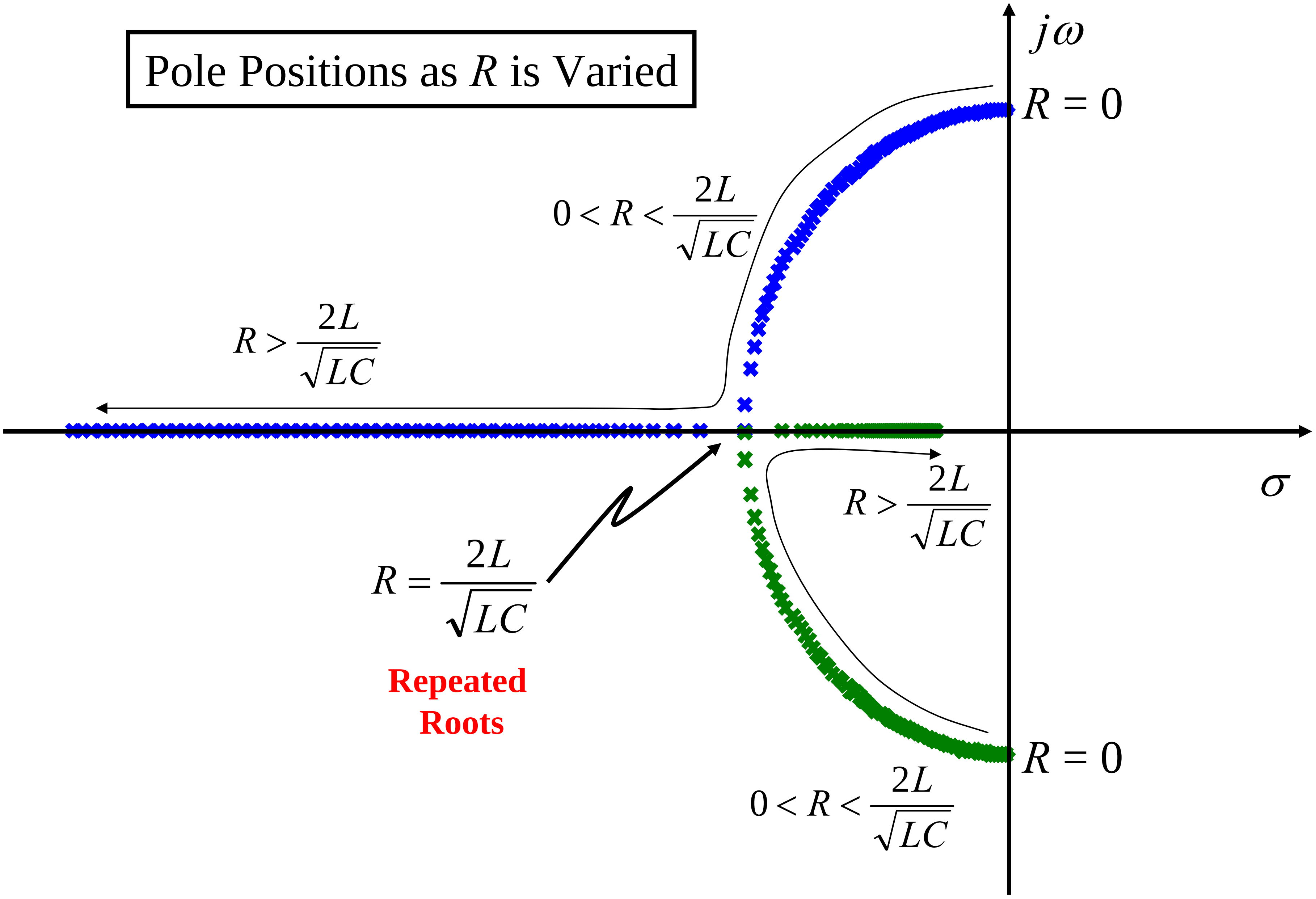
<u>Note:</u> For fixed L, C values we can drastically change the circuit's behavior by adjusting R



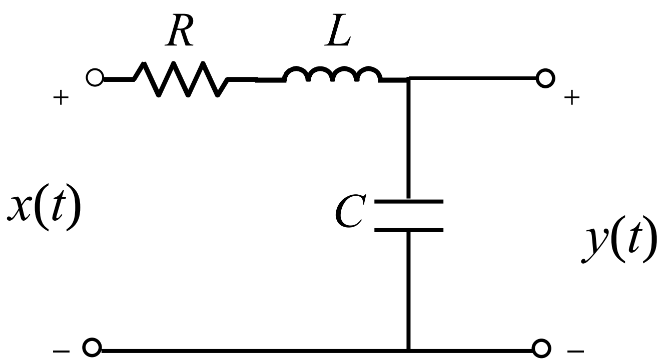
<u>Note:</u> A larger R value will more highly "damp" the oscillations

See the plot on the next page for pole positions as *R* is varied

Pole Positions as R is Varied



RLC Circuit: A 2nd Order Circuit



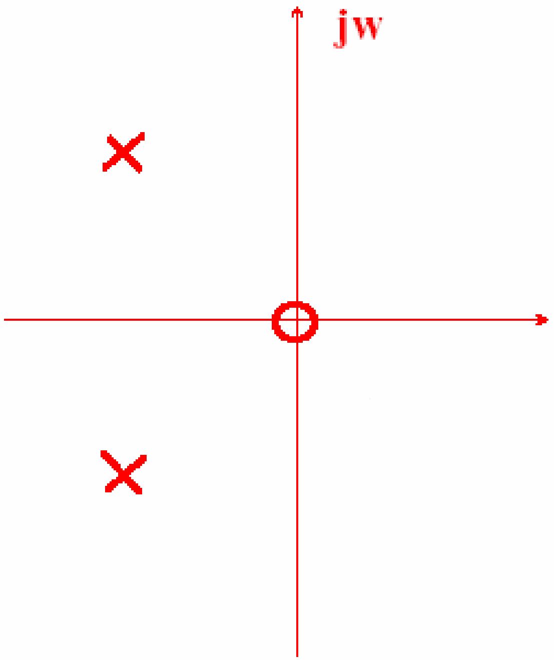
$$H(s) = \frac{1/LC}{s^2 + \left(R/L\right)s + 1/LC}$$

If $R < \frac{2L}{\sqrt{LC}}$ we get complex roots: $p_{1,2} = \sigma_o \pm j\omega_o$

Now we'll explore the effect of changing the component values to change the location of the poles for the complex poles case…

You've already seen the next 3 slides… they are repeated here as a reminder!!!

From the Pole-Zero Plots we can Visualize the TF function on the s-plane:



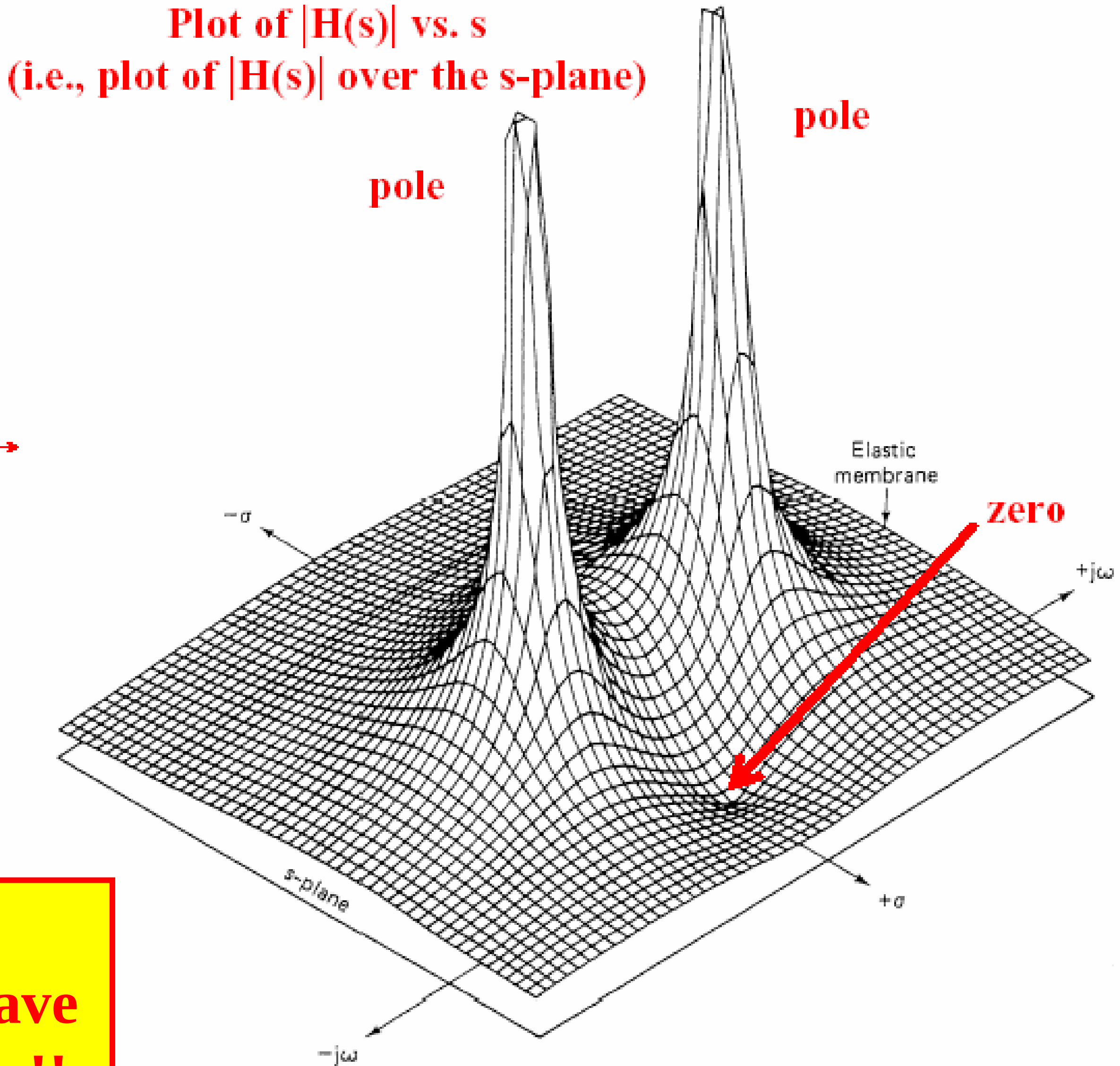


Fig. 9-4. Shape of elastic membrane for a pair of poles and a zero.

The circuit we're looking at doesn't have the zero at the origin!!

From our Visualization of the TF function on the s-plane we can see the Freq. Resp.:

To get the frequency response H(w) from the transfer function H(s) we replace s by jw… this is graphically equivalent to "cutting along the jw axis"

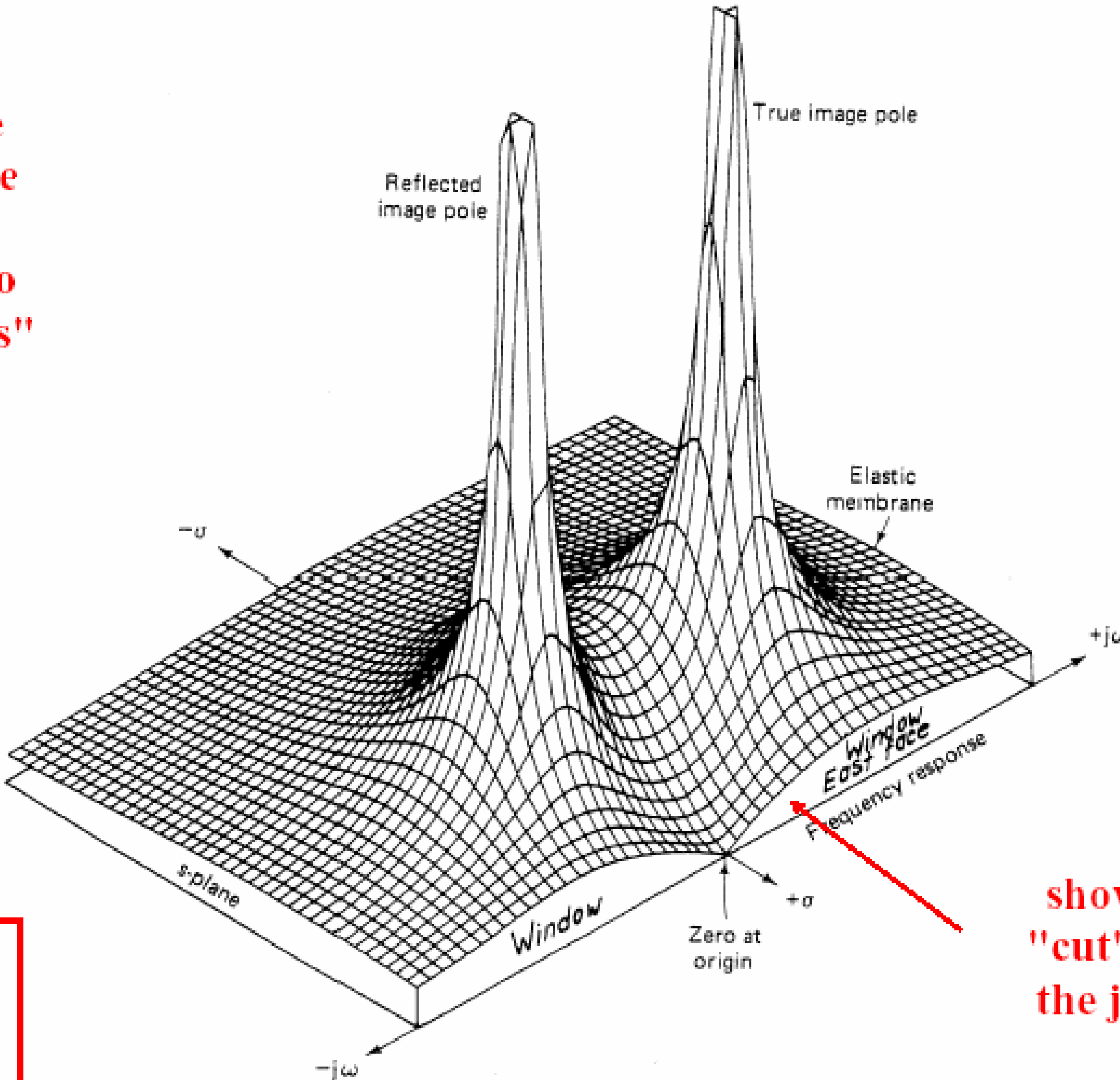
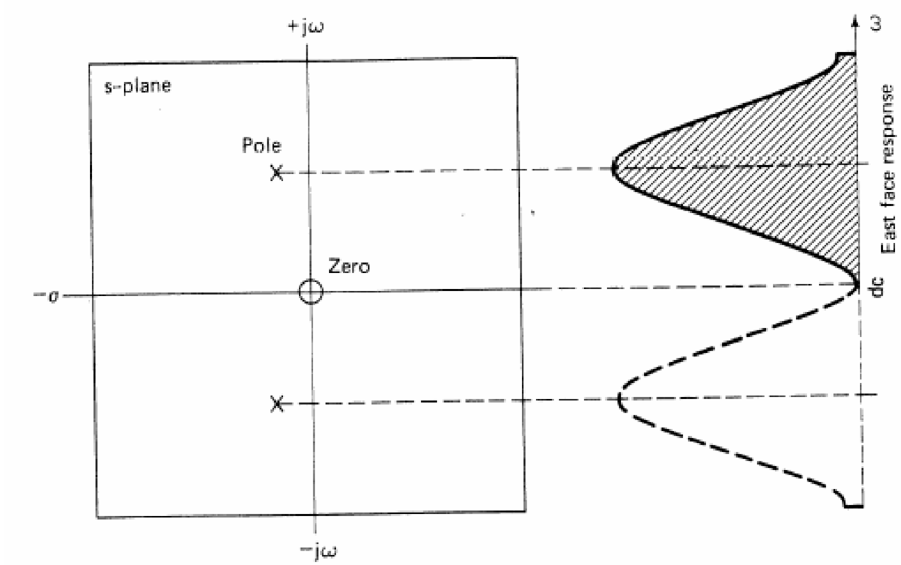


Fig. 9-5. Pole-zero diagram showing the east face.

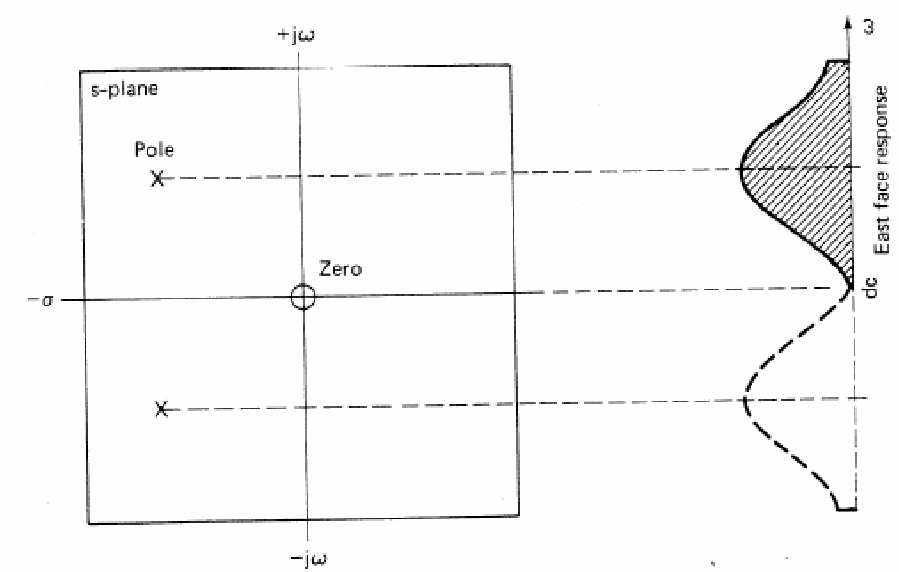
shows the "cut" along the jw axis

Plots from my favorite book on Op Amps: "Operational Amplifier: Characteristics and Applications" by Robert G. Irvine, Prentice-Hall, 1981

Can also look at a pole-zero plot and see the effects on Freq. Resp.



(a) Frequency response for pole close to jf axis



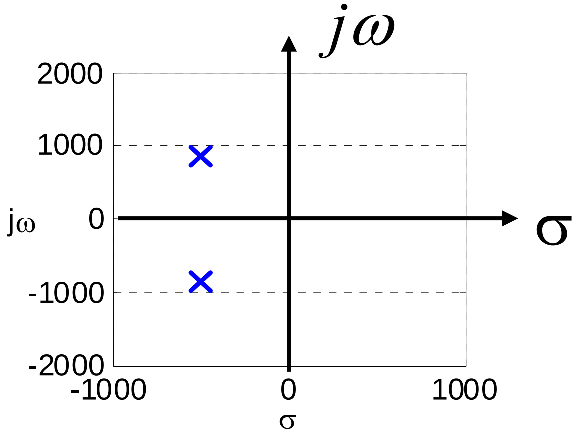
(b) Frequency response for pole far from jf axis

Fig. 9-6. Frequency response versus pole location.

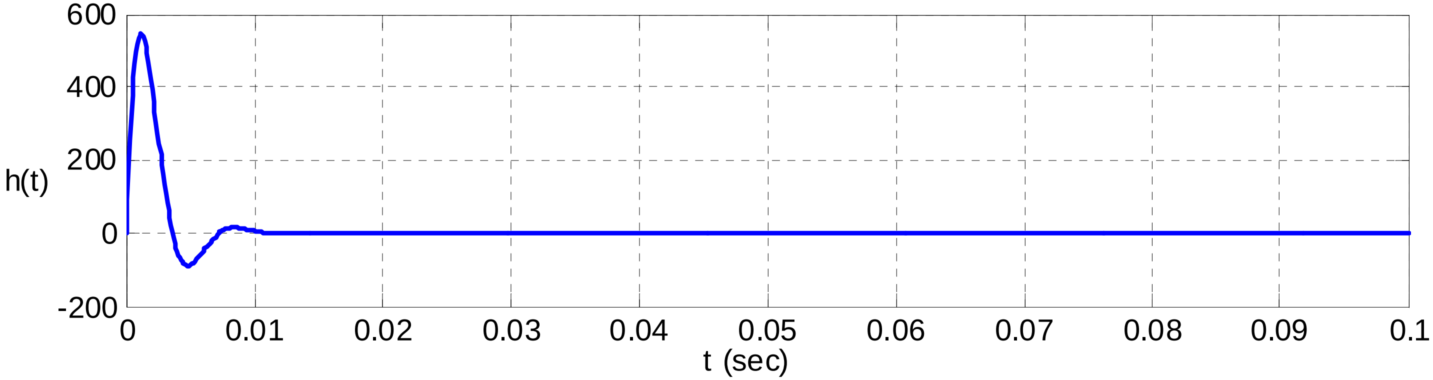
As the pole moves closer to the $j\omega$ axis it has a stronger effect on the frequency response $H(\omega)$. Poles close to the $j\omega$ axis will yield sharper and taller bumps in the frequency response.

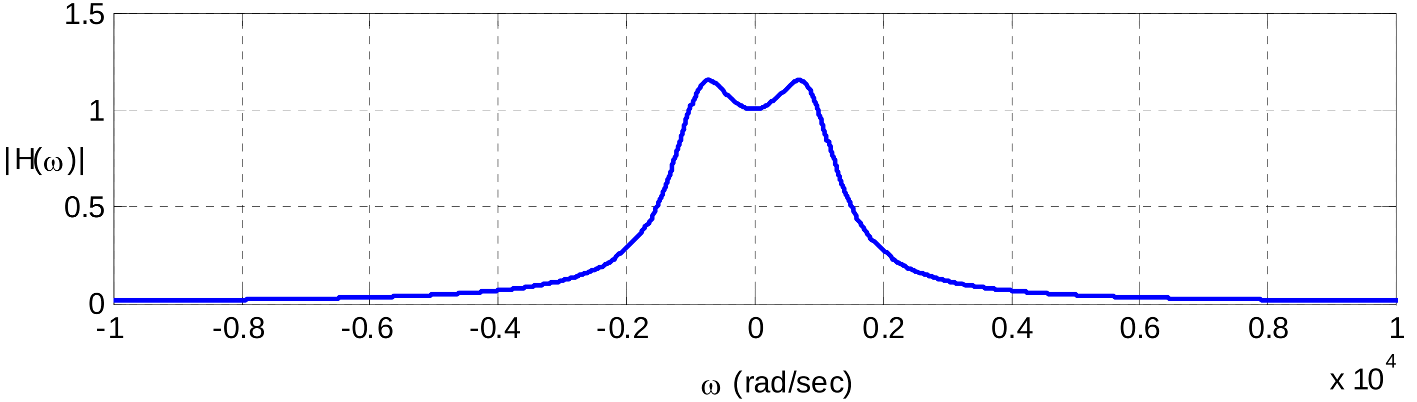
By being able to visualize what $|H(s)|$ will look like based on where the poles and zeros are, an engineer gains the ability to know what kind of transfer function is needed to achieve a desired frequency response… then through accumulated knowledge of electronic circuits (requires experience accumulated AFTER graduation) the engineer can devise a circuit that will achieve the desired effect.

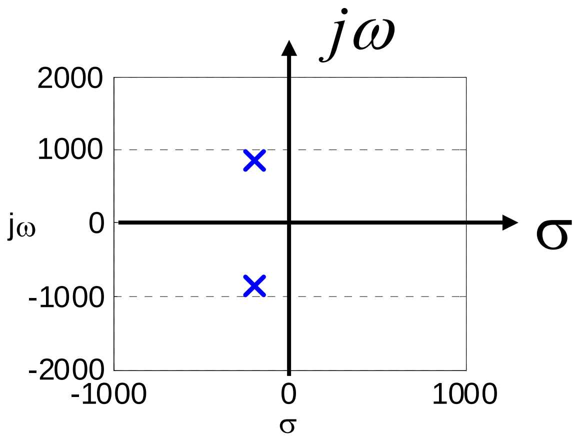
Effect of Changing σ_o



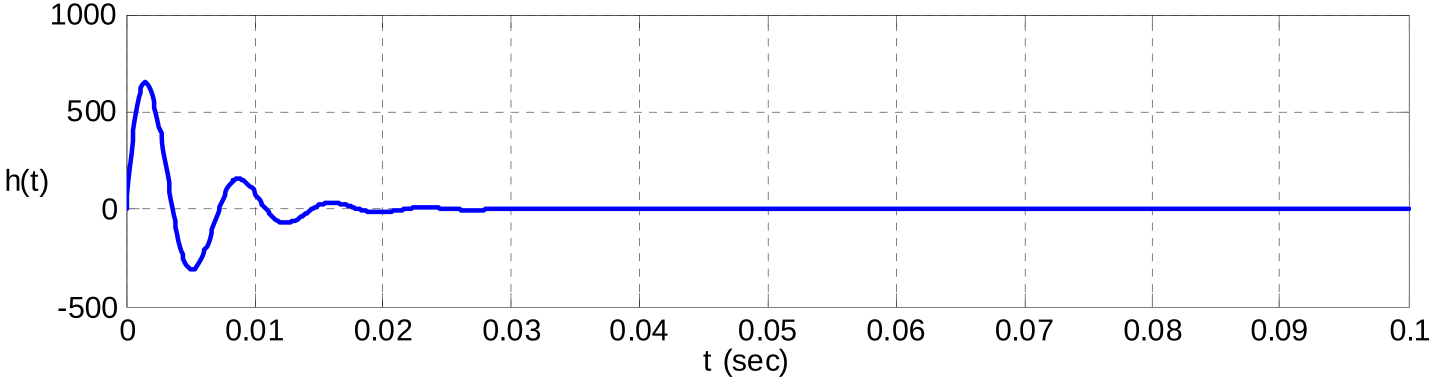
$\sigma_o = 500$ (1/sec)
$\omega_o = 866$ (rad/sec)

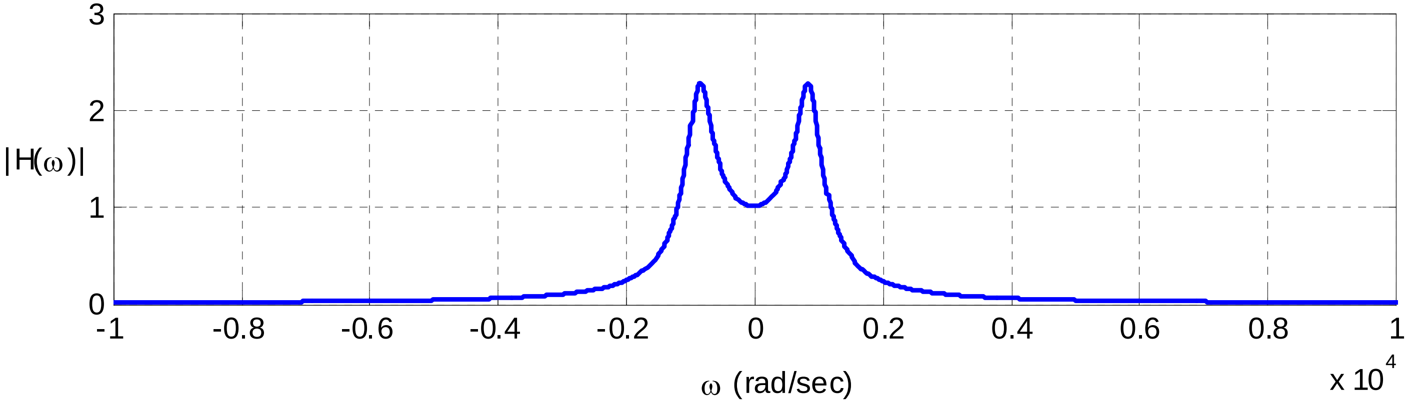


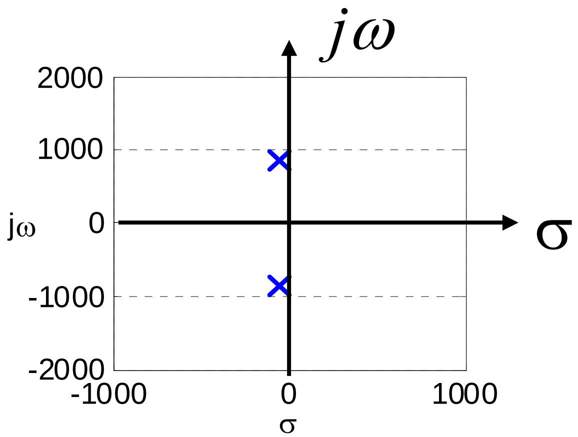




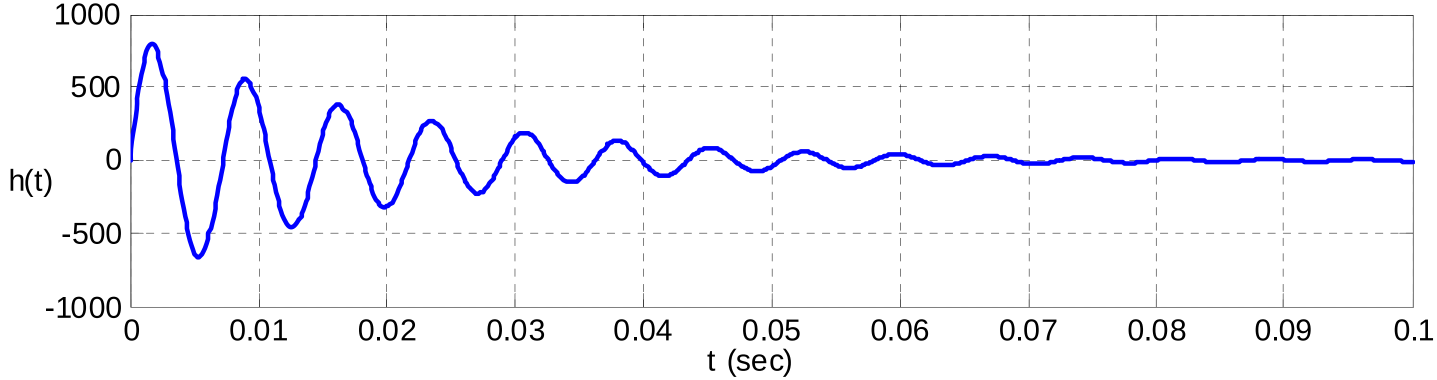
$\sigma_o = 200$ (1/sec)
$\omega_o = 866$ (rad/sec)

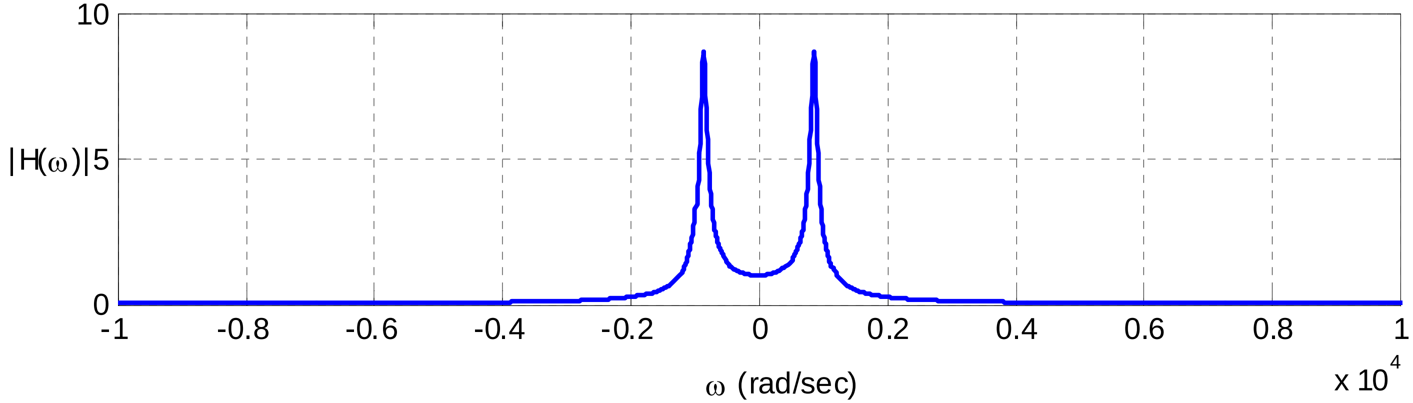




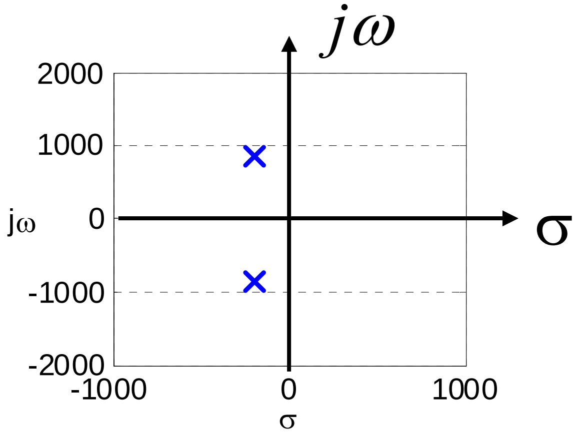


$\sigma_o = 50$ (1/sec)
$\omega_o = 866$ (rad/sec)

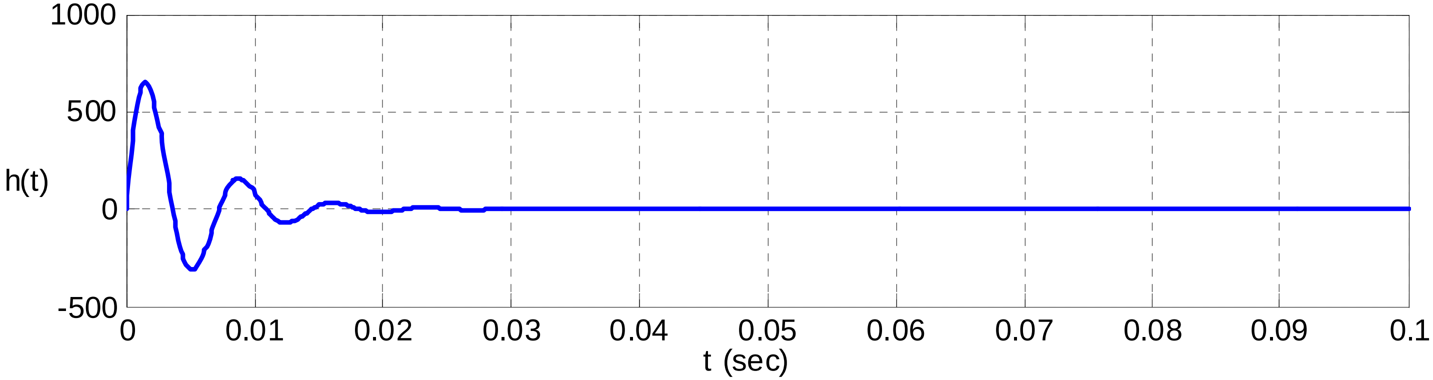


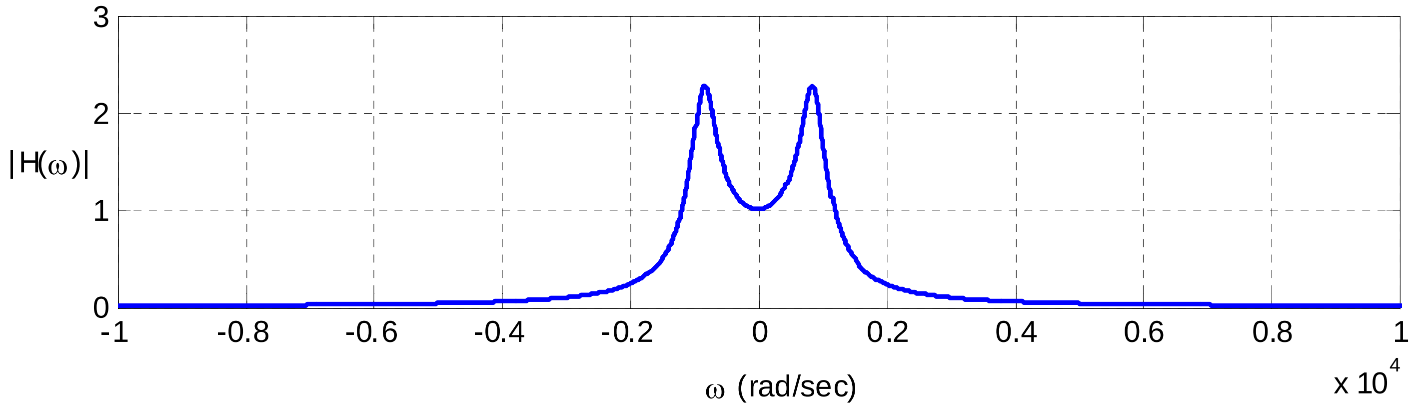


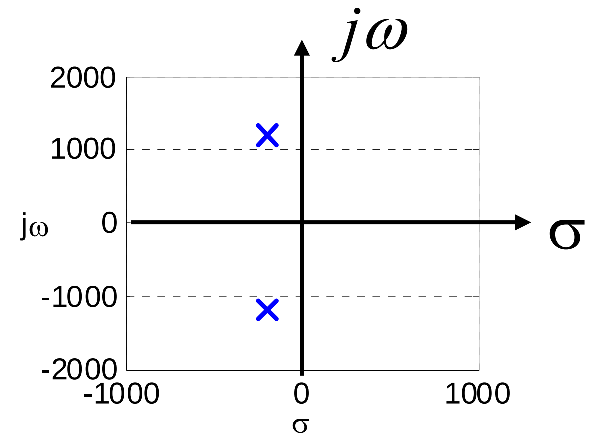
Effect of Changing ω_o



$\sigma_o = 200$ (1/sec)
$\omega_o = 866$ (rad/sec)

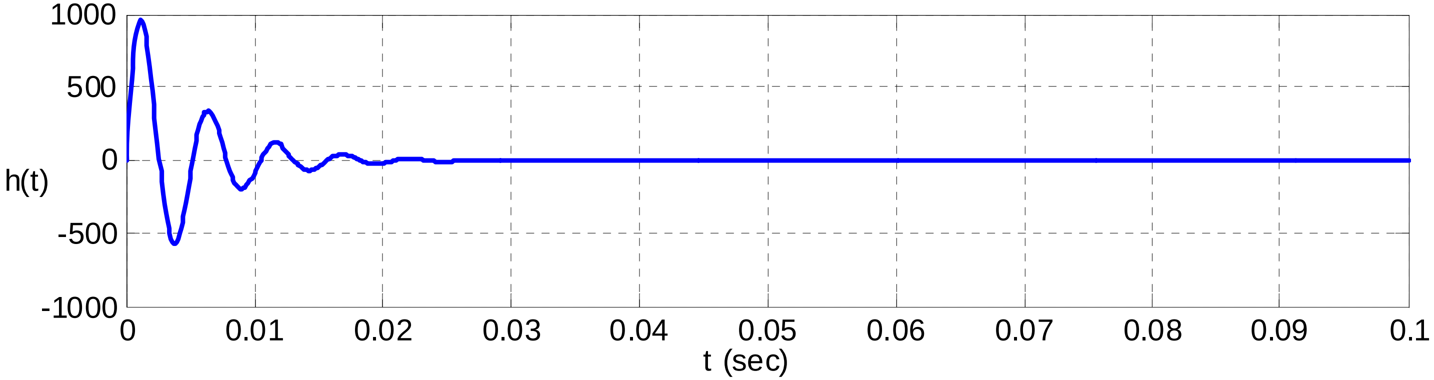


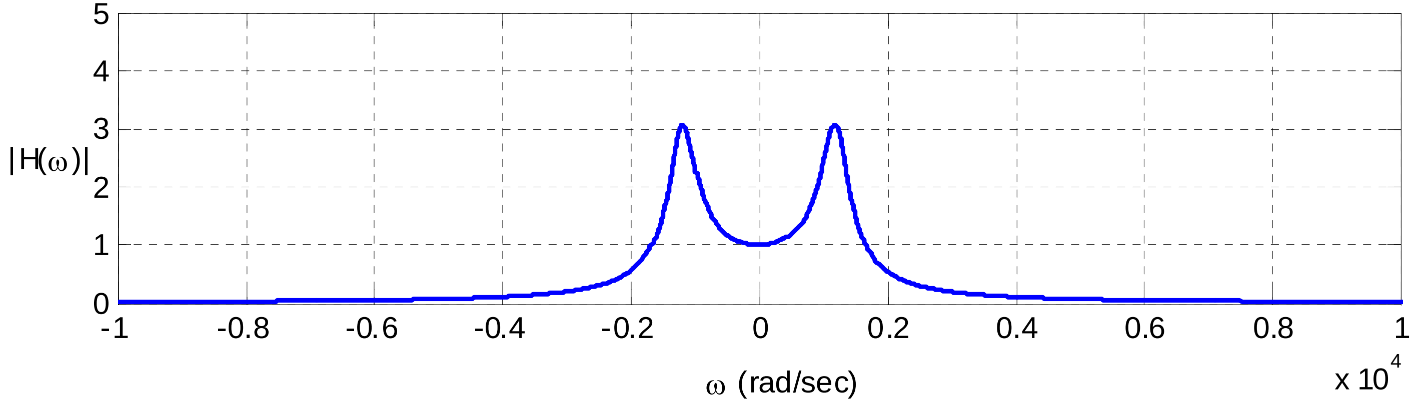


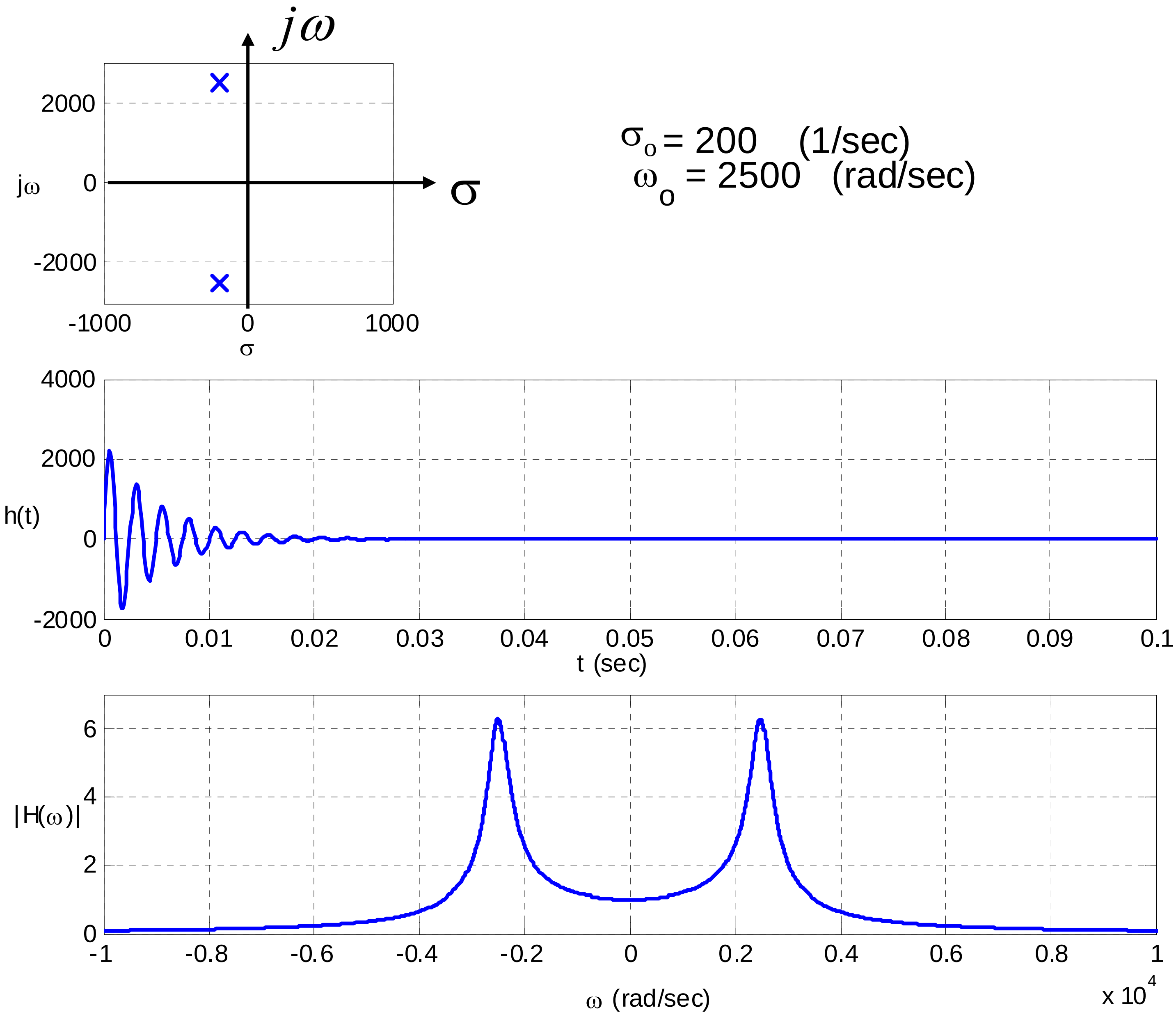


$\sigma_o = 200$ (1/sec)

$\omega_o = 1200$ (rad/sec)







jω
2000
0
-2000
-1000
0
1000
σ
σ
4000
2000
0
-2000
h(t)
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
t (sec)
6
4
2
0
|H(ω)|
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
x 10^4
ω (rad/sec)

σo = 200 (1/sec)
ωo = 2500 (rad/sec)