

Partial Fraction Expansion via MATLAB

The **“residue” function of MATLAB can be used to compute the partial fraction expansion (PFE)** of a ratio of two polynomials. This can be used for Laplace transforms or Z transforms, although we will illustrate it with Z transforms here.

The “residue” command gives three pieces of information:

- ▶ the residues are given in output vector r,
- ▶ the poles are given in output vector p,
- ▶ the so-called direct terms are given in output vector k.

When the order of the numerator polynomial is less than the order of the denominator polynomial there will be no direct terms.

When the order of the numerator equals the order of the denominator there will be one direct term;

When the order of the numerator is one greater than the order of the denominator there will be two direct terms.

When the order of the numerator is two greater than the order of the denominator there will be three direct terms; etc.

That is, **when the order of the numerator is p greater than the order of the denominator (with $p \geq 0$) there will be $p+1$ direct terms.**

The “residue” command requires two input vectors:

- ▶ one holding the coefficients of the numerator and
- ▶ one holding the coefficients of the denominator.

The right-most element in these vectors corresponds to the z^0 coefficient, the next element to the left is the z^1 coefficient, etc., until you reach the highest power; if a power is not present it has a zero coefficient.

It is easiest to explain how to use “residue” by giving examples.

Note: We will be **expanding “ $H(z)$ divided by z ”** like is done when trying to find the inverse Z transform – the **division by z is a trick** used to ensure that after PFE we get a form for which it is usually easy to look up the inverse ZT on a table.

Note also that **this division does not change the denominator** of what we are expanding – this is important because **the denominator sets the major characteristic** of the corresponding time signal (e.g., the denominator of a system’s transfer function determines the system’s characteristic modes).

Thus, every example will be done using $G(z)$ **where $G(z) = H(z)/z$** .

Example #1: No Direct Terms, No Repeated Roots, No Complex Roots

$$G(z) = H(z)/z = \frac{3z-1}{z^2-3z+2}$$

In this case there will be no direct terms because the numerator order is lower than the denominator order.

The **numerator vector** is [3 -1]

The **denominator vector** is [1 -3 2].

The **command and its result** is:

» [r,p,k]=residue([3 -1],[1 -3 2])

r =

5
-2

p =

2
1

k =

[]

Note that k is empty (denoted by []) showing that there are no direct terms.

The **numbers in r are the residues** – they are the numbers that go in the numerator of each term.

The **numbers in p are the poles** – they are the numbers subtracted from z to form the denominators of the terms.

We have a **pole at $z = 2$ with a residue of 5** and....

...a **pole at $z = 1$ with a residue of -2**; thus:

$$G(z) = \frac{3z-1}{z^2-3z+2} = \frac{5}{z-2} + \frac{-2}{z-1}$$

and then

$$H(z) = zG(z) = \frac{z(3z-1)}{z^2-3z+2} = \frac{5z}{z-2} + \frac{-2z}{z-1}$$

Example #2: One Direct Term, No Repeated Roots, No Complex Roots

$$G(z) = H(z)/z = \frac{2z^2 + 3z - 1}{z^2 - 3z + 2}$$

» [r,p,k]=residue([2 3 -1],[1 -3 2])

r =
13
-4

p =
2
1

k =
2

Thus, we have:

a pole at: 2 w/ residue 13 and

a pole at: 1 w/ residue -4;

.... But in this case we **also have a single direct term of 2**, which just adds a term of 2 to the expansion:

$$G(z) = \frac{2z^2 + 3z - 1}{z^2 - 3z + 2} = \frac{13}{z - 2} + \frac{-4}{z - 1} + 2$$

Thus...

$$H(z) = zG(z) = \frac{z(2z^2 + 3z - 1)}{z^2 - 3z + 2} = \frac{13z}{z - 2} + \frac{-4z}{z - 1} + 2z$$

Example #3: Two Direct Terms, No Repeated Roots, No Complex Roots

$$G(z) = H(z)/z = \frac{7z^3 + 2z^2 + 3z - 1}{z^2 - 3z + 2}$$

» [r,p,k]=residue([7 2 3 -1],[1 -3 2])

$$r = \begin{array}{l} 69 \\ -11 \end{array}$$

$$p = \begin{array}{l} 2 \\ 1 \end{array}$$

$$k = \begin{array}{l} 7 \quad 23 \end{array}$$

This **gives two terms in the vector k** so we know we **have two direct terms: 7z and 23.**

Note: if there were M elements in vector k the left most term would be the coefficient of a z^M order direct term.) Thus, we have

$$G(z) = \frac{7z^3 + 2z^2 + 3z - 1}{z^2 - 3z + 2} = \frac{69}{z - 2} + \frac{-11}{z - 1} + 7z + 23$$

Thus...

$$H(z) = \frac{69z}{z - 2} + \frac{-11z}{z - 1} + 7z^2 + 23z$$

Example #4: No Direct Terms, A Double Root, No Complex Roots

$$G(z) = H(z)/z = \frac{2z^2 + 3z - 1}{z^3 - 5z^2 + 8z - 4} = \frac{2z^2 + 3z - 1}{(z - 1)(z - 2)^2}$$

» [r,p,k]=residue([2 3 -1],[1 -5 8 -4])

r =

-2.0000
13.0000
4.0000

p =

2.0000
2.0000
1.0000

k =

[]

Note that **p has two elements listed as 2.0000**: that means that there is a **double root**.

Note also that there are **two residues for these poles** – namely, -2 and 13:

- ▶ **First residue** is for the **first-order term for the double root**
- ▶ **Second residue** is for the **second-order term for the double root**.

Thus, the expansion is

$$G(z) = \frac{2z^2 + 3z - 1}{z^3 - 5z^2 + 8z - 4} = \frac{2z^2 + 3z - 1}{(z-1)(z-2)^2} = \frac{-2}{z-2} + \frac{13}{(z-2)^2} + \frac{4}{z-1}$$

Note, in general: if there were an **Mth order pole** we would have **a term for each order up to and including M** for that pole.

Such terms (after multiplying by z to undo the division by z done before PFE when computing the inverse Z transform) **are inverse Z transformed using #8 - #10 in Table 11.1 of the book.**

Example #5: No Direct Terms, No Repeated Roots, Two Complex Roots

$$G(z) = H(z)/z = \frac{2z^2 + 3z - 1}{z^3 - 3z^2 + 4z - 2} = \frac{2z^2 + 3z - 1}{(z-1)(z^2 - 2z + 2)}$$

$$= \frac{2z^2 + 3z - 1}{(z-1)(z-[1+j])(z-[1-j])}$$

This shows how the given **denominator polynomial** is **factored into three poles, two of them complex**.

Note that the **two complex poles are conjugates of each other** – this is a necessary condition for the coefficients of the polynomial to be real valued.

Thus, for the cases we are interested in, **we will always have complex poles occur in “conjugate pairs”**.

The **use of “residue” is no different here**,
but **what we do with the output is a little different**:

- ▶ **combine the conjugate pair terms into a single second order term** – which would then be inverse Z transformed using one of the #11 - #12 lines in Table 11.1 of the book (after multiplying by z to undo the division by z done before PFE when computing the inverse Z transform).
- ▶ Of course, **#12b in the table allows us to also go directly** from the two conjugate terms – we’ll do it this way here.

» [r,p,k]=residue([2 3 -1],[1 -3 4 -2])

r =

-1.0000 - 3.5000i
 -1.0000 + 3.5000i
 4.0000

p =

1.0000 + 1.0000i
 1.0000 - 1.0000i
 1.0000

k =

[]

$$G(z) = \frac{2z^2 + 3z - 1}{z^3 - 3z^2 + 4z - 2} = \frac{-1 - j3.5}{z - (1 + j)} + \frac{-1 + j3.5}{z - (1 - j)} + \frac{4}{z - 1}$$

Now, to **illustrate the inverse Z transform of such conjugate pair terms**, we **multiply each term in the above to by z to get H(z)**. Thus, the two conjugate terms become:

$$\frac{z(-1 - j3.5)}{z - (1 + j)} + \frac{z(-1 + j3.5)}{z - (1 - j)}$$

Converting these into polar form gives:

$$\frac{3.6401e^{-j1.8491}z}{z - (\sqrt{2}e^{j\pi/4})} + \frac{3.6401e^{+j1.8491}z}{z - (\sqrt{2}e^{-j\pi/4})}$$

Thus, $r = 2 * 3.6401 = 7.2802$, $\theta = -1.8491$, $|\gamma| = \sqrt{2}$ $\beta = \pi/4$. Thus, these two terms inverse transform to

$$7.2802(\sqrt{2})^k \cos((\pi/4)k - 1.8491)u[k]$$