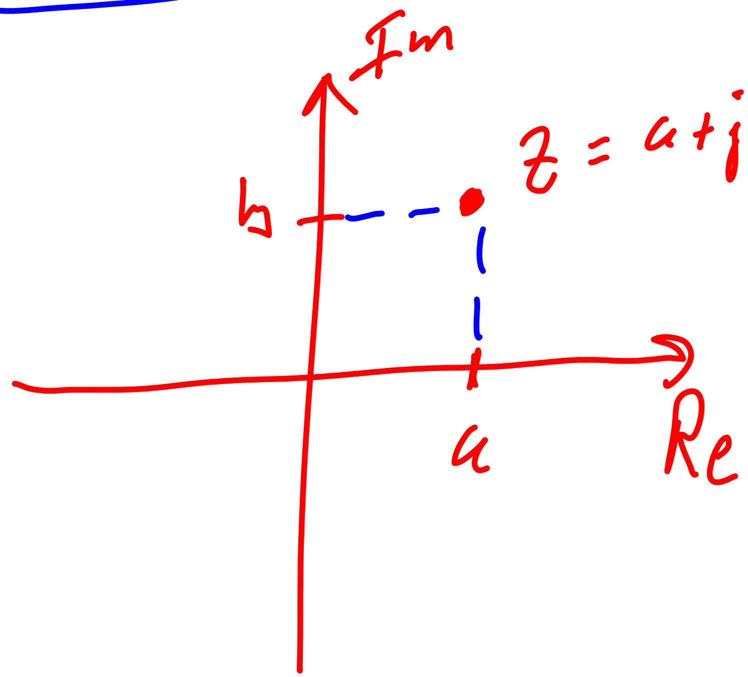
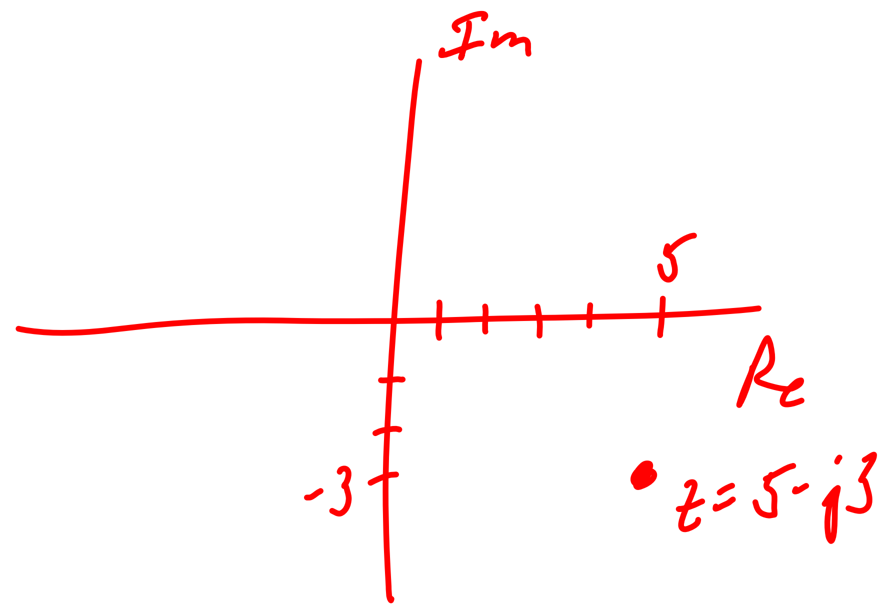
Define Imaginary # j $\qquad \boxed{j = \sqrt{-1}}$ $\qquad$ A

$$x^2 + 1 = 0 \qquad x^2 = -1$$

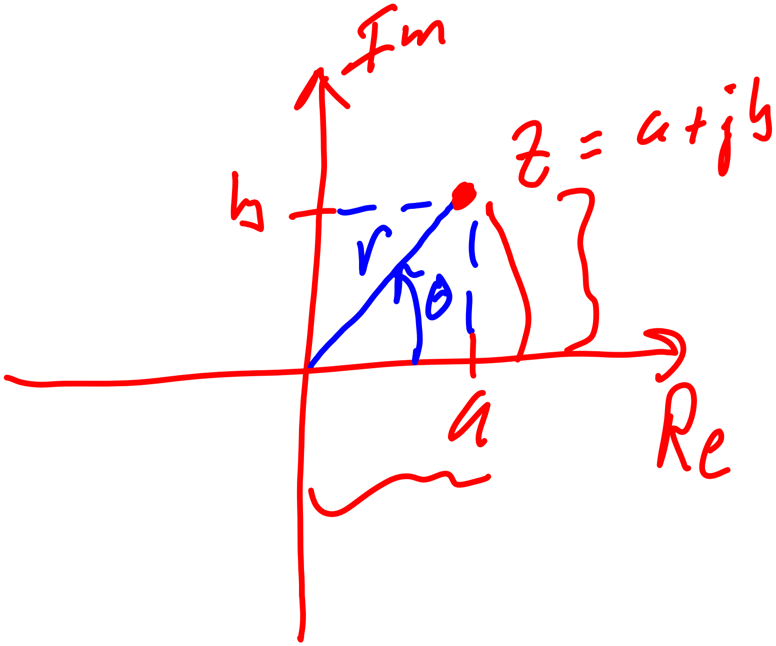
$$\boxed{z = \underline{a} + j\underline{b}}$$ Rectangular Form



$z = 5 - j3$



B



θ: Angle Convention

- measured as positive in a counterclockwise direction
- θ is in radians

r: Magnitude Convention

$r \geq 0$

$z = 5e^{j\cancel{45^\circ}}$

$z = 5e^{j\pi/4}$

Polar Form

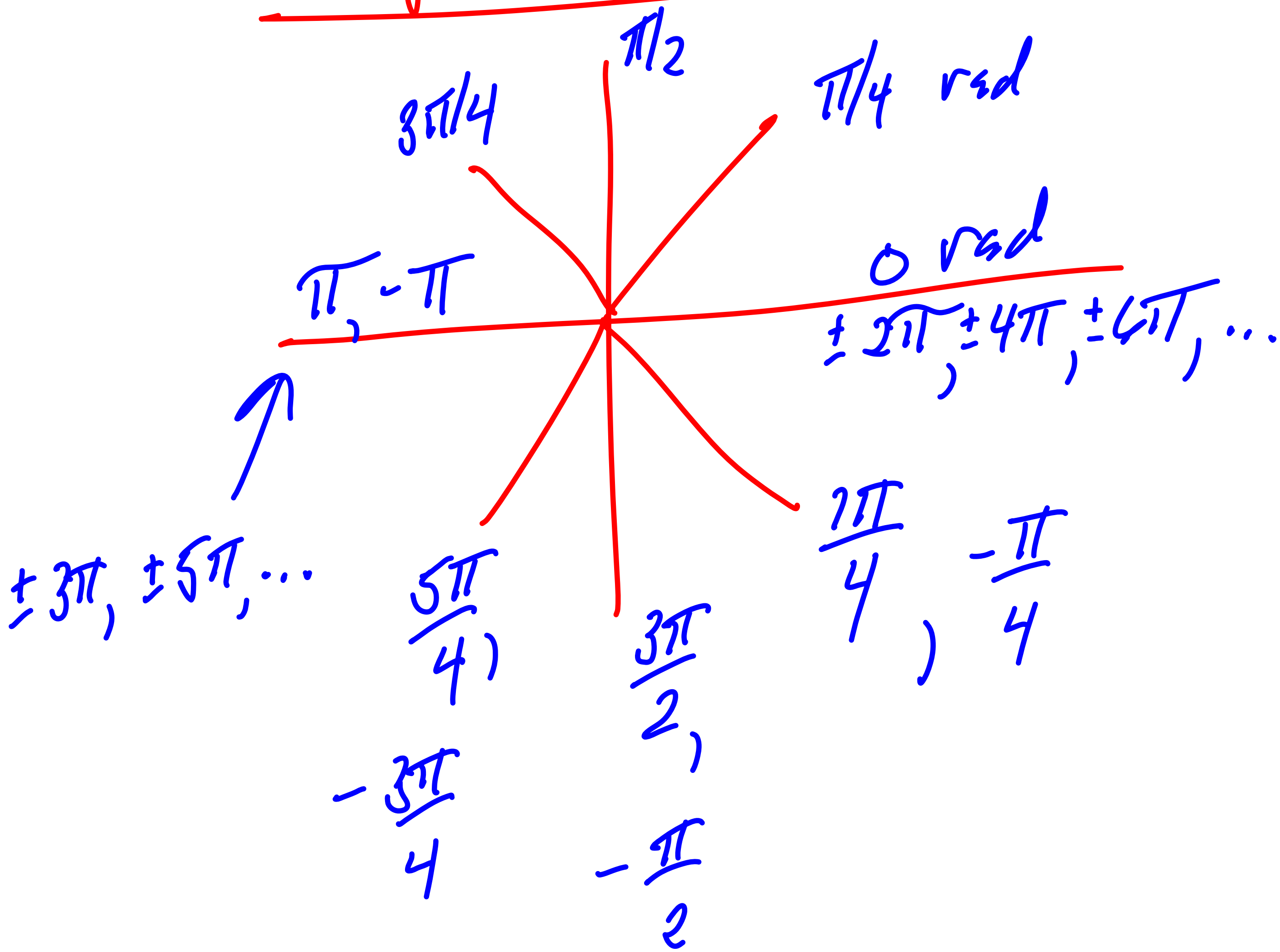
2 Equivalent Versions

Version #1: $z = r\angle\theta$

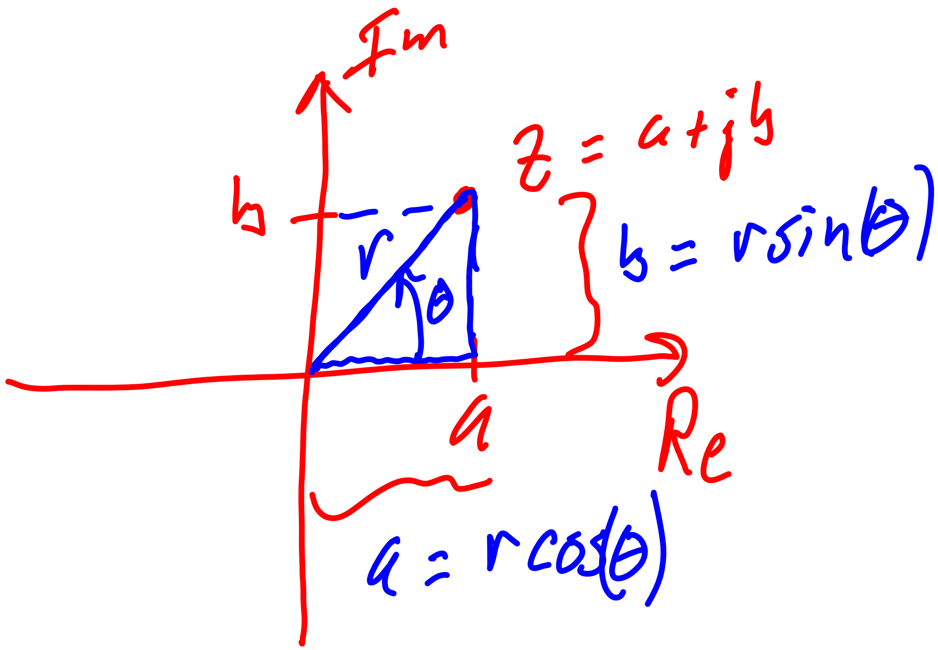
Version #2: $\boxed{z = re^{j\theta}}$ ← Preferred Version

Angles in Radians

C



Converting Between Rect. & Polar Form (1)



Pythagorean

$$r = \sqrt{a^2+b^2}$$

Trig: $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

From Polar to Rect

Given $z = re^{j\theta}$

Convert $z = r\cos\theta + jr\sin\theta$

From Rect. to Polar

Given $z = a+jb$

Convert $z = \sqrt{a^2+b^2}\, e^{j\tan^{-1}\left(\frac{b}{a}\right)}$

Warning: Calculators may incorrectly compute $\tan^{-1}\left(\frac{b}{a}\right)$

See Later

Euler's Formulas

E

$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$

$$\boxed{\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$\boxed{e^{-j\theta} = \cos(\theta) - j\sin(\theta)}$$

$$\boxed{\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

Add Them

$\Downarrow$

$$e^{j\theta} + e^{-j\theta} = 2\cos(\theta)$$

Manipulation Rules

F

Addition: $\underbrace{(a+jb)}_{z_1} + \underbrace{(c+jd)}_{z_2} = (a+c) + j(b+d)$

Subtraction: $\underbrace{(a+jb)}_{z_1} - \underbrace{(c+jd)}_{z_2} = (a-c) + j(b-d)$

Do using Rect. Form

6

Mulplication:

$$\boxed{\left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = \left(r_1 r_2\right) e^{j(\theta_1+\theta_2)}}$$

$$\boxed{a^x a^y = a^{x+y}}$$

Division:

$$\boxed{\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{j(\theta_1-\theta_2)}}$$

$$\boxed{\frac{a^x}{a^y} = a^{x-y}}$$

Use Polar Form

$$z_1 = r_1 e^{j\theta_1} \Rightarrow \boxed{\frac{1}{z_1} = \frac{1}{r_1} e^{-j\theta_1}}$$

$$\boxed{|z_1 z_2| = |z_1|\,|z_2|}$$

$$\boxed{\angle(z_1 z_2) = \angle z_1 + \angle z_2}$$

H

$$\underbrace{(r_1 e^{j\theta_1})(r_2 e^{j\theta_2})}_{z_1 \times z_2} = \underbrace{(r_1 r_2)}\, e^{j\underbrace{(\theta_1+\theta_2)}}$$

$$\boxed{\left|\frac{z_1}{z_1}\right| = \frac{|z_1|}{|z_2|}}$$

$$\boxed{\angle\left(\frac{z_1}{z_2}\right) = \angle z_1 - \angle z_2}$$

$$z^n = (re^{j\theta})^n = r^n (e^{j\theta})^n = r^n e^{jn\theta}$$

$$(a^x)^n = a^{nx}$$

$$\boxed{z^n = r^n e^{jn\theta}}$$

$$\boxed{z^{1/n} = r^{1/n} e^{j\theta/n}}$$

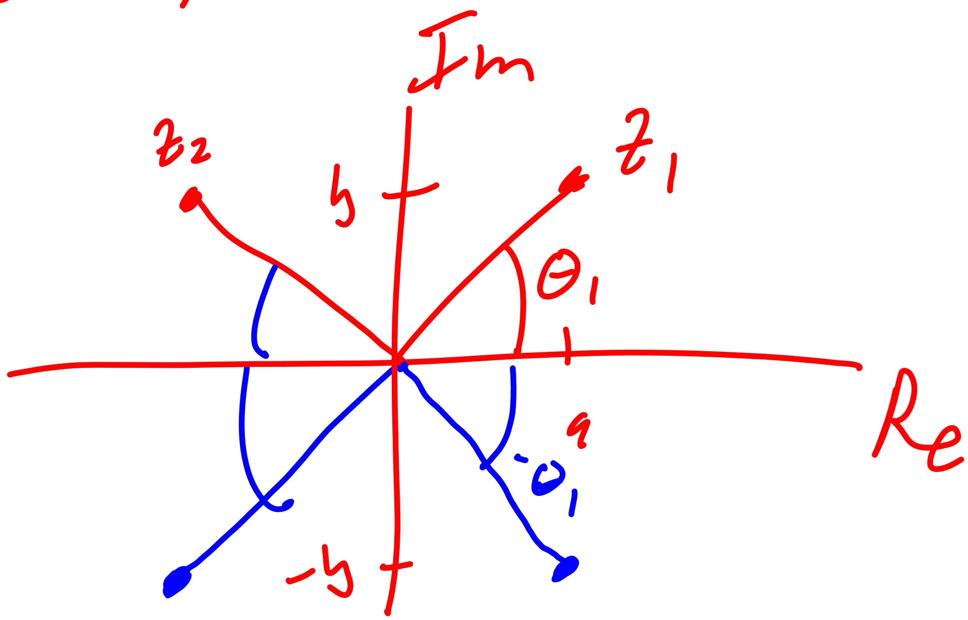
Conjugate of z

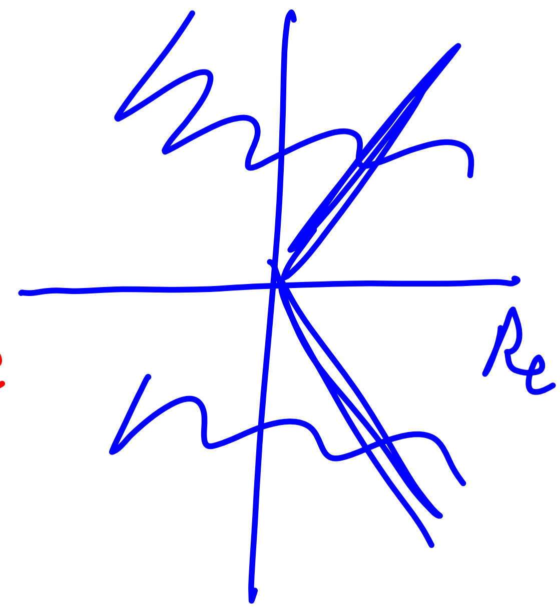
(J)

Let $z = a + jb = re^{j\theta}$

Define $z^* = a - jb = re^{-j\theta}$ ← Conjugate

(or $\bar{z}$)





Prop. of Conj.

$z = a + jb$

$(a+jb) + (a-jb) = 2a$

$$\boxed{\begin{aligned} &1.\ z + z^* = 2\,Re\{z\} \\ &2.\ z\,z^* = a^2 + b^2 = |z|^2 \end{aligned}}$$

$|z|^2 = a^2 + b^2$

Trick

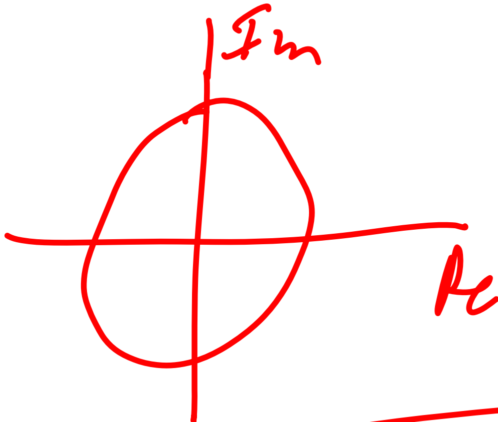
$$\frac{z_1}{z_2}\,\frac{z_2^*}{z_2^*} = \frac{z_1 z_2^*}{|z_2|^2}$$

$(a+jb)(a-jb)$

$= a^2 + \cancel{jab} - \cancel{jab} + b^2$

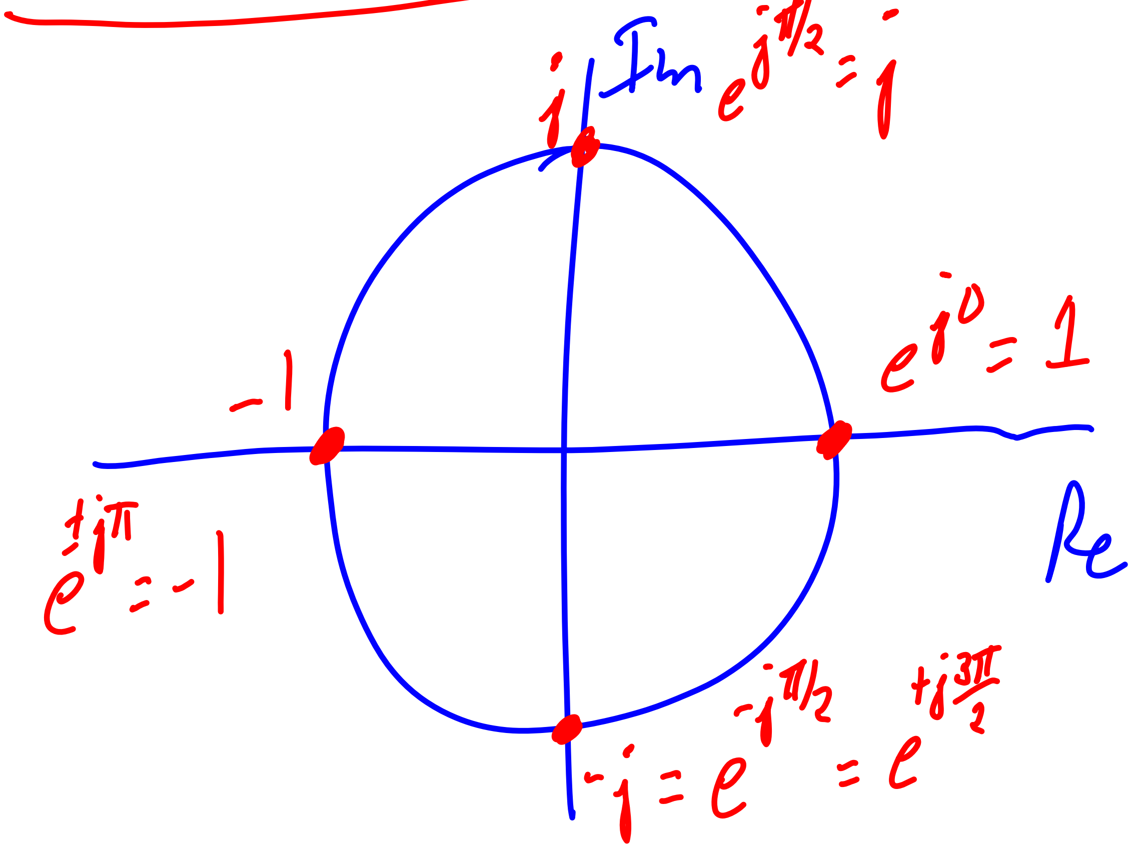
Unit Circle & Special Complex #'s

2



U.C. = Set of z w/ $|z| = 1$

$$U.C. = \{ e^{j\theta} \mid -\pi \le \theta \le \pi \}$$



Know These

$$e^{\pm jn\pi} = \begin{cases} -1, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$$

$$e^{\pm jn2\pi} = 1 \quad \forall\, n \text{ integer}$$

$$e^{jn\frac{\pi}{2}} = \begin{cases} j, & n = 1, 5, 9, \dots \\ -j, & n = 3, 7, 11, \dots \\ 1, & n = 0, 4, 8, \dots \\ -1, & n = 2, 6, 10, \dots \end{cases}$$

Examples

Ⓜ

① $z = -4 - j3$ convert to polar

$$|z| = \sqrt{(-4)^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

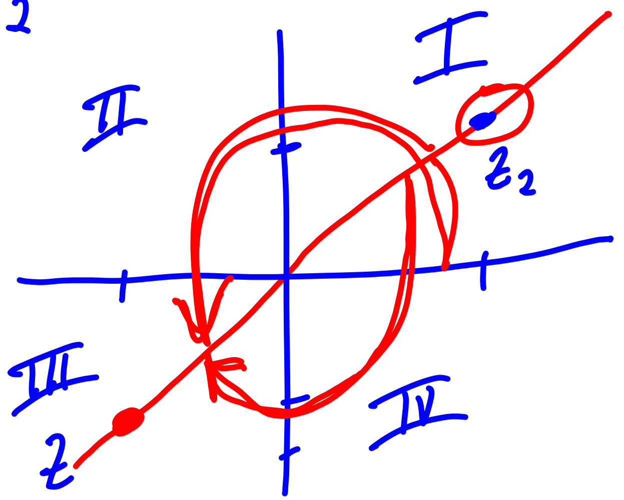
$$\angle z = \tan^{-1}\left(\frac{-3}{-4}\right) = \tan^{-1}(0.75) = 0.64 - \underbrace{\pi}_{\text{calc. conv. factor}}$$

$$= -2.5 \text{ rad}$$

$$\Rightarrow \boxed{z = 5e^{-j2.5}}$$

$$\parallel$$

$z_2 = 4 + j3 \rightarrow \tan\left(\frac{3}{4}\right)$



Calc. will always give the answer in Quadrants I & IV

(2) Convert from Polar to Rect. N

$$z = 3e^{j\pi/4}$$

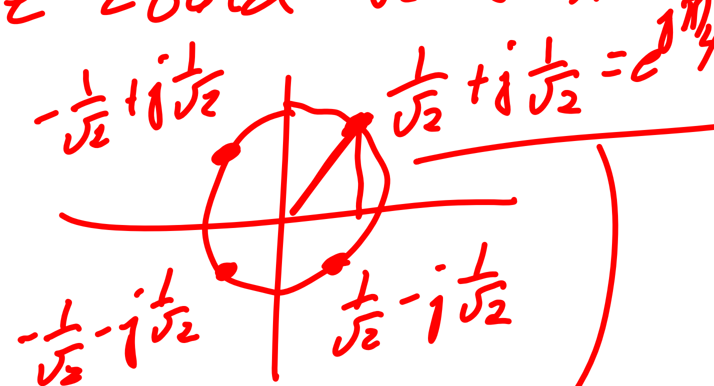
Could blindly Apply these:

$$z = r\cos\theta + j r\sin\theta$$

$$= \underbrace{3\cos\left(\tfrac{\pi}{4}\right)} + \underbrace{j3\sin\left(\pi/4\right)}$$

$$= \sim + j \sim$$

Or we could be smart.



$$z = 3e^{j\pi/4} = 3\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$$

$$\boxed{z = \frac{3}{\sqrt{2}} + j\frac{3}{\sqrt{2}}}$$

③ Write $je^{-j\pi/2}$ in polar form. ⓪

Q: Isn't it already in P.F.?

Blind Application

$$(j)\left(e^{-j\pi/2}\right)$$

$z_1 \quad z_2$

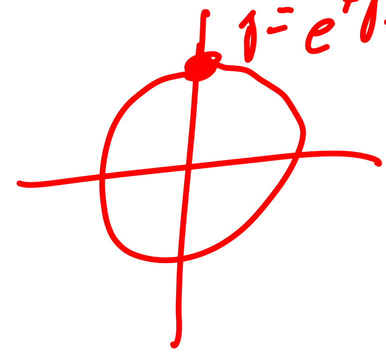
$$|z_1| = \sqrt{0^2+1^2} = 1$$

$$\angle z_1 = \tan^{-1}\left(\frac{1}{0}\right) = ?$$

Smart Visualization

$$(j)\left(e^{-j\pi/2}\right)$$

$j = e^{+j\frac{\pi}{2}}$



$$= \left(e^{+j\pi/2}\right)\left(e^{-j\pi/2}\right)$$

$$= 1e^{j(\pi/2-\pi/2)} = 1\underbrace{e^{j0}}_{=1}$$

$$je^{-j\pi/2} = 1e^{j0} = \underline{1}$$

4. Write $je^{j\pi/4}$ in rect. form. P

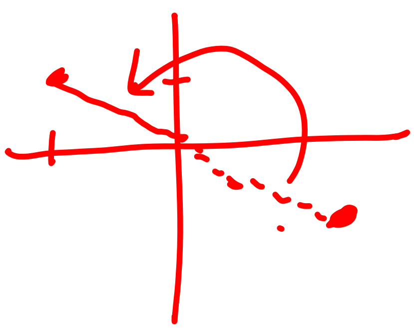
$$je^{j\pi/4} = j\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) = \frac{j}{\sqrt{2}} + \underbrace{j^2}_{=-1}\frac{1}{\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

---

5. Write $\dfrac{2+j3}{-3+j2}$ in polar form

Convert N & D to polar: $\dfrac{\cancel{3.61}\,e^{j0.98}}{\cancel{3.61}\,e^{j2.55}} = 1e^{j(0.98-2.55)}$



$$\approx e^{-j1.57} = e^{-j\pi/2}$$

6 $2e^{j6\pi}(1+j)$ convert into polar Q

$e^{j6\pi} = e^{j3(2\pi)} = 1$

apply $\sqrt{2^2+2^2}$, $\tan^{-1}\left(\frac{2}{2}\right)$

$= 2\times 1(1+j) = 2+j2$

$\hookrightarrow = \sqrt{2}\, e^{j\pi/4}$

know $e^{j\pi/4} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

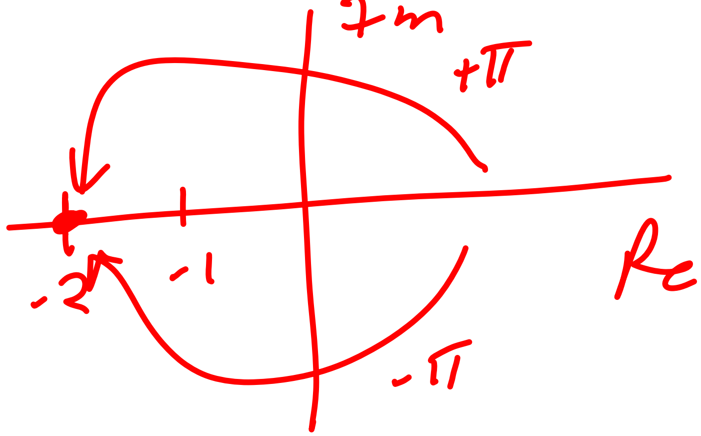
$= 2\left(\sqrt{2}\, e^{j\pi/4}\right) = 2\sqrt{2}\, e^{j\pi/4}$

7) $\angle 2j e^{j\pi/2} = ?$

A R

Know $e^{j\pi/2} = j \Rightarrow \angle 2j \cdot j = \angle -2 = ?$

$\angle -2 = \pm\pi$ rad



$\angle +\text{Real} = 0$

$\angle -\text{Real} = \pm\pi$

Alternate Way

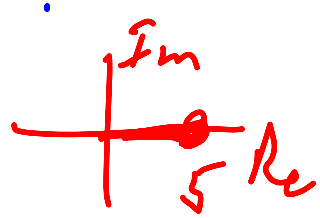
$\angle 2e^{j\pi/2}e^{j\pi/2} = \angle \underbrace{2e^{j\pi}} = \pi$ rad

8) $\angle -5 = ?$

$\angle -5 = \pm\pi$ rad

$\angle 5 = ?$

$\angle 5 = 0$ rad



(9) Find $\angle (-1)^n e^{jn\pi/4}$ for n integer (5)

$$\angle (-1)^n e^{jn\pi/4} = \underbrace{\angle (-1)^n}_{=\,?} + \underbrace{\angle e^{jn\pi/4}}_{=\,n\pi/4} = \begin{cases} n\pi/4, & n \text{ even} \\ \pi + n\pi/4, & n \text{ odd} \end{cases}$$