I-1: Review of DT Signal & System Concepts

Notational Conventions (P-1.2)

Sets of Numbers: R = real numbers
C = complex numbers
Z = integers (…, -3, -2, -1, 0, 1, 2, 3, …)

Signals: $x(t)$, $y(t)$ = continuous-time (CT) signals
$x[n]$, $y[n]$ = discrete-time (DT) signals

Proakis & Manolakis don't follow this

DT Convolution: $\{x * y\}[n] = \sum_{m=-\infty}^{\infty} x[m]y[n-m]$

Proakis & Manolakis don't follow this

DT Circular Convolution: $\{x \circledast y\}[n] = \sum_{m=0}^{N-1} x[m]y[(n-m) \bmod N] \quad 0 \le n \le N-1$

Recall: We don't want Circular convolution – it is what we get if we try to implement Convolution in the frequency domain by multiplying DFT's (but not DTFT's)
But… we can fix it so that it works…. We use proper zero-padding!!!

Transforms (P-2)

Proakis & Manolakis don't follow this Notation

Fourier Transform for CT Signals

$$X^F(\omega) = \{Fx\}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^F(\omega)e^{j\omega t}d\omega$$

Fourier Transform for DT Signals

$$X^{\mathrm{f}}(\theta) = \{Fx\}(\theta) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\theta n}$$

Discrete
Time
Fourier
Transform

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^{\mathrm{f}}(\theta)e^{j\theta n}d\theta$$

Set $z = e^{j\theta}$

Z Transform for DT Signals

$$X^{\mathrm{z}}(z) = \{Zx\}(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Inverse ZT done using partial fractions & a ZT table

Discrete Fourier Transform for DT Signals

Discrete **F**ourier **T**ransform

$$X^{\mathrm{d}}[k] = \{Dx\}[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \qquad k = 0,1,2,\ldots,N-1$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{d}}[k]e^{j2\pi kn/N} \qquad n = 0,1,2,\ldots,N-1$$

<u>Fourier Transforms for DT Signals (P-2.7,P-4, PM-4, PM-7)</u>

<u>DTFT</u> = "In Head/On Paper" tool for analysis & design

<u>DFT</u> = "In Computer" tool for <u>Implementation</u>/<u>Analysis&Design</u>
(In both cases you generally strive to make the DFT behave like DTFT)

- DFT for Implementation:
  - Compute DFT of signal to be processed and manipulate DFT values
  - May or may not need zero-padding (depends on application)
- DFT for Analysis & Design:
  - Compute DFT of given filter's TF to see its frequency response
  - Generally use zero-padding to get fine grid to get approximate DTFT

<u>DFT/DTFT Properties</u> you **<u>Must</u> <u>Know</u>:**

- Periodicity…………………. (DFT&DTFT both periodic in FD)
  …………………. (IDFT gives periodic TD signal)
- Time Shift………………….. (Time Shift in TD ⇔ Linear Phase Shift in FD)
- Frequency Shift……………. (Mult. by complex sine in TD ⇔ Freq Shift in FD)
- Convolution Theorem ……… (Conv. in Time Domain ⇔ Mult. in Freq Domain)
- Multiplication Theorem…….. (Mult. in Time Domain ⇔ Conv. in Freq Domain)
- Parseval's Theorem…………. (Sum of Squares in TD = Sum of Squares in FD)

<u>Summation Rules (P-1.3)</u>

Make sure you are familiar with rules for dealing with summations, as shown in Section 1.3 of Porat.

In addition, a particular summation formula that shows up often is the **"Geometric Summation"**:

$$\sum_{n=N_1}^{N_2-1} \alpha^n = \begin{cases} \dfrac{\alpha^{N_1} - \alpha^{N_2}}{1-\alpha}, & \text{if } \alpha \neq 1 \\[2ex] N_2 - N_1, & \text{if } \alpha = 1 \end{cases}$$

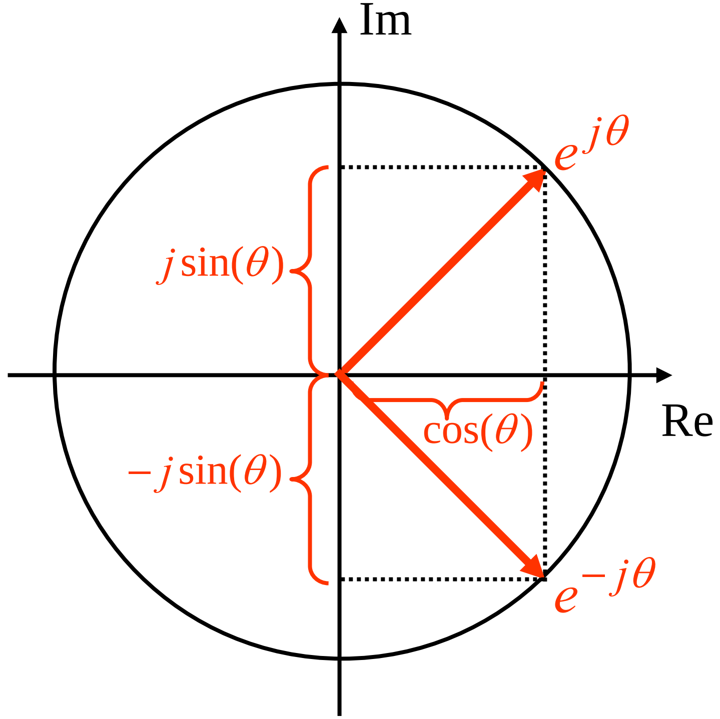
<u>Special Case: $N_1 = 0$</u>

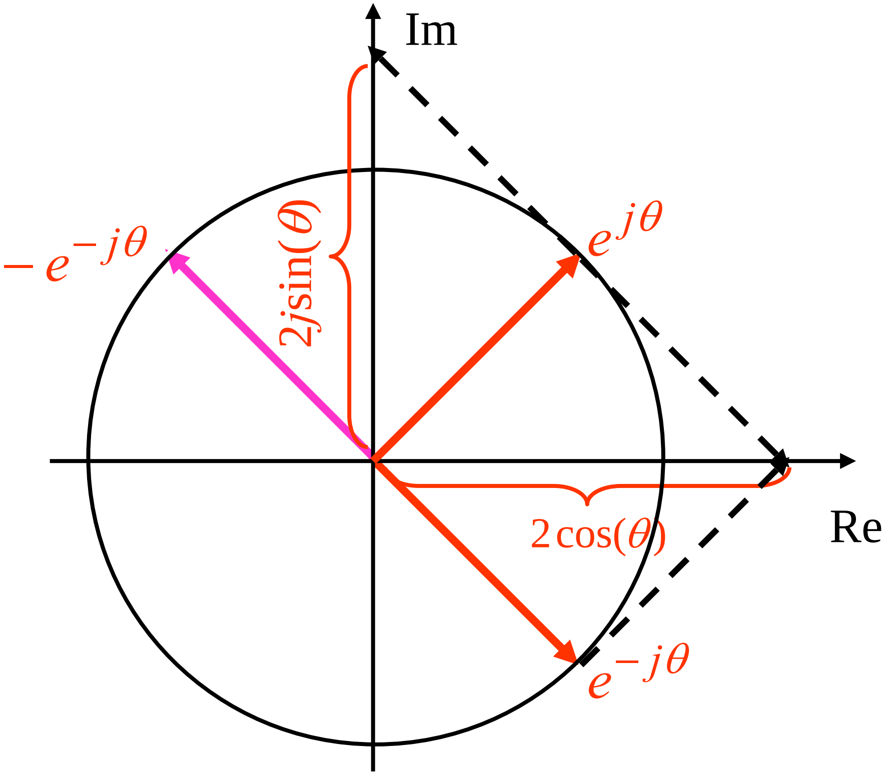
$$\sum_{n=0}^{N_2-1} \alpha^n = \begin{cases} \dfrac{1 - \alpha^{N_2}}{1-\alpha}, & \text{if } \alpha \neq 1 \\[2ex] N_2, & \text{if } \alpha = 1 \end{cases}$$

Euler's Formulas

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$





Discrete-Time Processing (P-2, PM-5)

$x[n] \quad X^{f}(\theta) \quad X^{z}(z) \longrightarrow \boxed{h[n] \quad H^{f}(\theta) \quad H^{z}(z)} \longrightarrow y[n] = h[n]*x[n] \quad Y^{f}(\theta) = H^{f}(\theta)\, X^{f}(\theta) \quad Y^{z}(z) = H^{z}(z)\, X^{z}(z)$

$h[n]$ = system's "Impulse Response"

It is response of system to input of $\delta[n]$

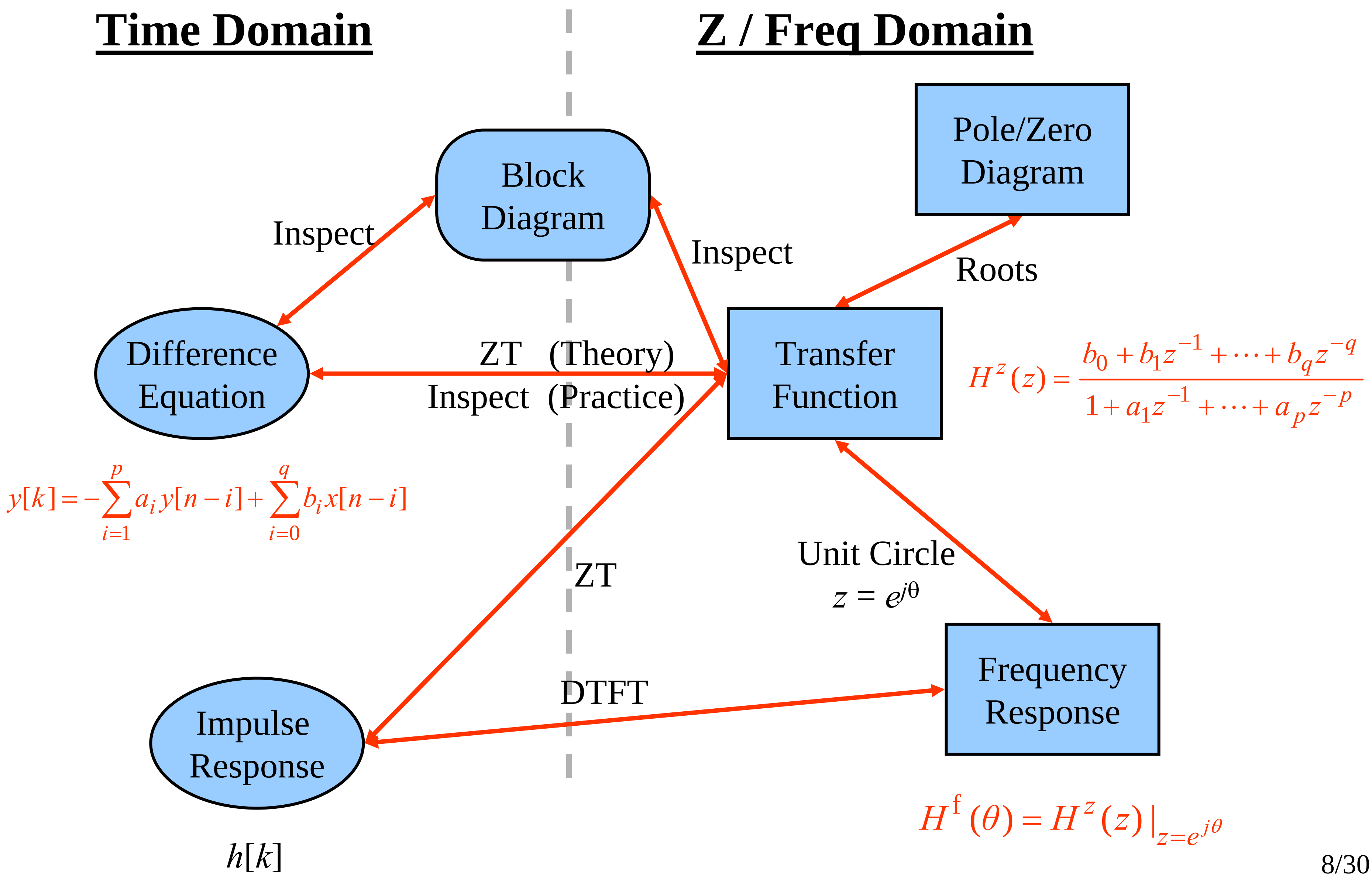
$H^{f}(\theta)$ = system's "Frequency Response"

It is DTFT of impulse response

$H^{z}(z)$ = system's "Transfer Function"

It is ZT of impulse response

Discrete-Time System Relationships



Poles and Zeros of Transfer Function

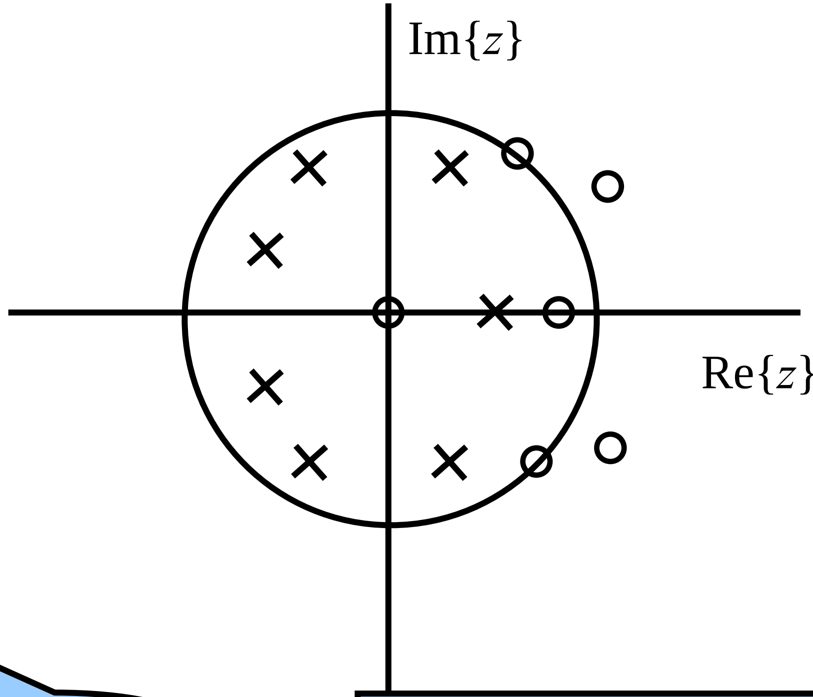
Consider Here Only p>q

$$H^z(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}}$$

$$= \frac{z^{p-q} z^q (b_0 + b_1 z^{-1} + \cdots + b_q z^{-q})}{z^p (1 + a_1 z^{-1} + \cdots + a_p z^{-p})}$$

$$= \frac{z^{p-q} (b_0 z^q + b_1 z^{q-1} + \cdots + b_q)}{z^p + a_1 z^{p-1} + \cdots + a_p}$$

$$= \frac{z^{p-q} (z-\beta_1)(z-\beta_2)\cdots(z-\beta_q)}{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_p)}$$



In this case
$p-q$ zeros at $z=0$

Numerator Roots define "zeros"
Denominator Roots define "poles"

Poles must be inside unit circle for stability

Poles & zeros must occur in Complex Conjugate pairs to give real-valued TF coefficients

IIR and FIR Filters

$$\boxed{H^z(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}}}$$

FIR: Finite Impulse Response… $h[n]$ has only finite many nonzero values

Filter is FIR: If $a_i = 0$ for $i = 1, 2, \ldots, p$ (**It is IIR otherwise**)

$$\boxed{H^z(z) = b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}}$$

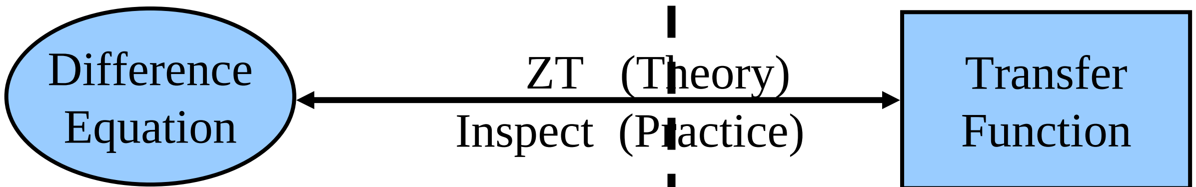
$$\boxed{h[n] = \begin{cases} b_n, & n = 0,1,\ldots,q \\ 0, & otherwise \end{cases}}$$

TF coefficients *are* the impulse response values!!

Example System

Time Domain

Z / Freq Domain



Input-Output Form

$$\boxed{y(n) - \alpha y(n-1) = \beta x(n)}$$

$$\boxed{Y(z)\left[1 - \alpha z^{-1}\right] = \beta X(z)}$$

Recursion Form

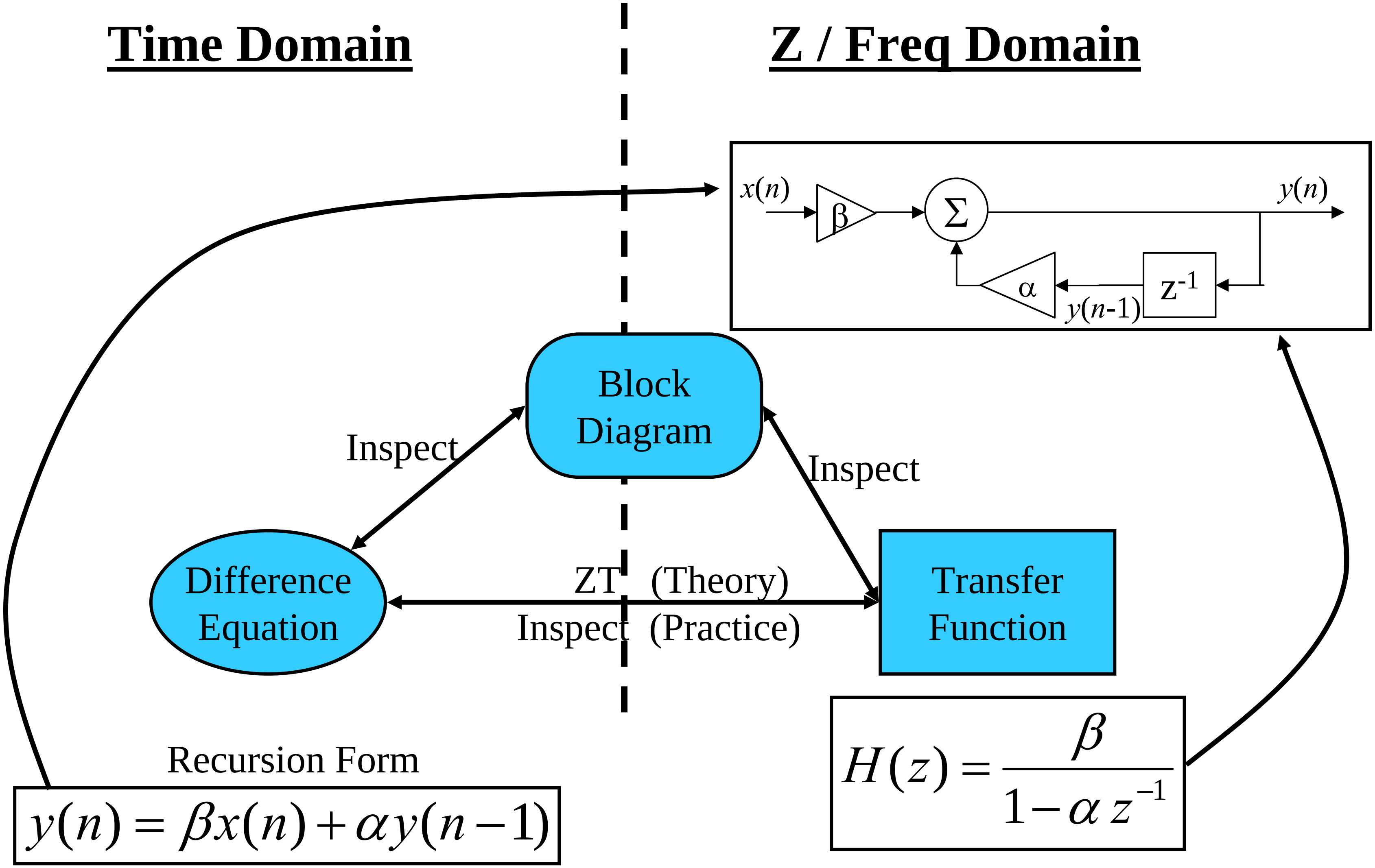
$$\boxed{y(n) = \beta x(n) + \alpha y(n-1)}$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\beta}{1 - \alpha z^{-1}}}$$

Example System (cont.)

Time Domain

Z / Freq Domain

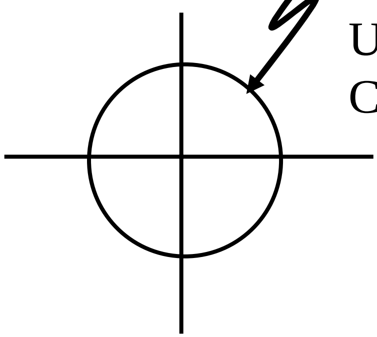


Example System (cont.)

Z / Freq Domain

$$z = e^{j\Omega}$$
$$z^{-1} = e^{-j\Omega}$$

On Unit Circle



$$H(z) = \frac{\beta}{1-\alpha z^{-1}}$$

$$z^{-1} = e^{-j\Omega}$$

$$H(\Omega) = \frac{\beta}{1-\alpha e^{-j\Omega}}$$

$$\Omega \in (-\pi, \pi]$$

Transfer Function

Unit Circle

Frequency Response

Example System (cont.)

Plotting Transfer Function

$$H(\Omega) = \frac{\beta}{1-\alpha e^{-j\Omega}} = \frac{\beta}{1-\left[\alpha\cos(\Omega) - j\alpha\sin(\Omega)\right]}$$

$$= \frac{\beta}{\left[1-\alpha\cos(\Omega)\right] + j\alpha\sin(\Omega)}$$

$$\left|H(\Omega)\right| = \frac{\beta}{\sqrt{\left[1-\alpha\cos(\Omega)\right]^2 + \left[\alpha\sin(\Omega)\right]^2}}$$

$$\angle H(\Omega) = -\tan^{-1}\left[\frac{\alpha\sin(\Omega)}{1-\alpha\cos(\Omega)}\right]$$

Euler's Equation

Group into Real & Imag

Standard Eqs for Mag. & Angle

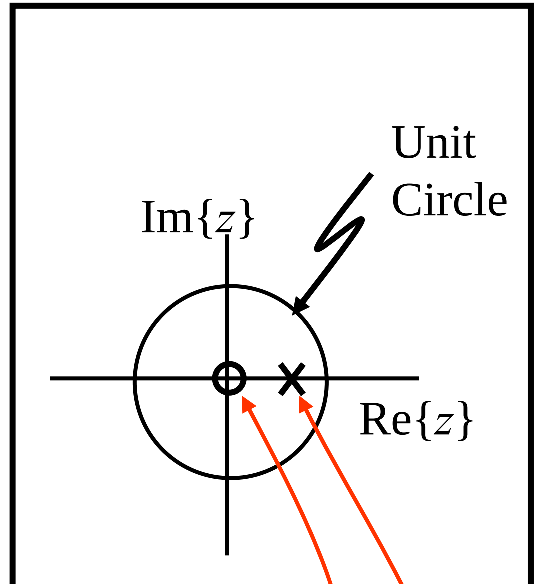
Example System (cont.)

Z/Freq Domain

Pole/Zero Diagram

Roots

Transfer Function



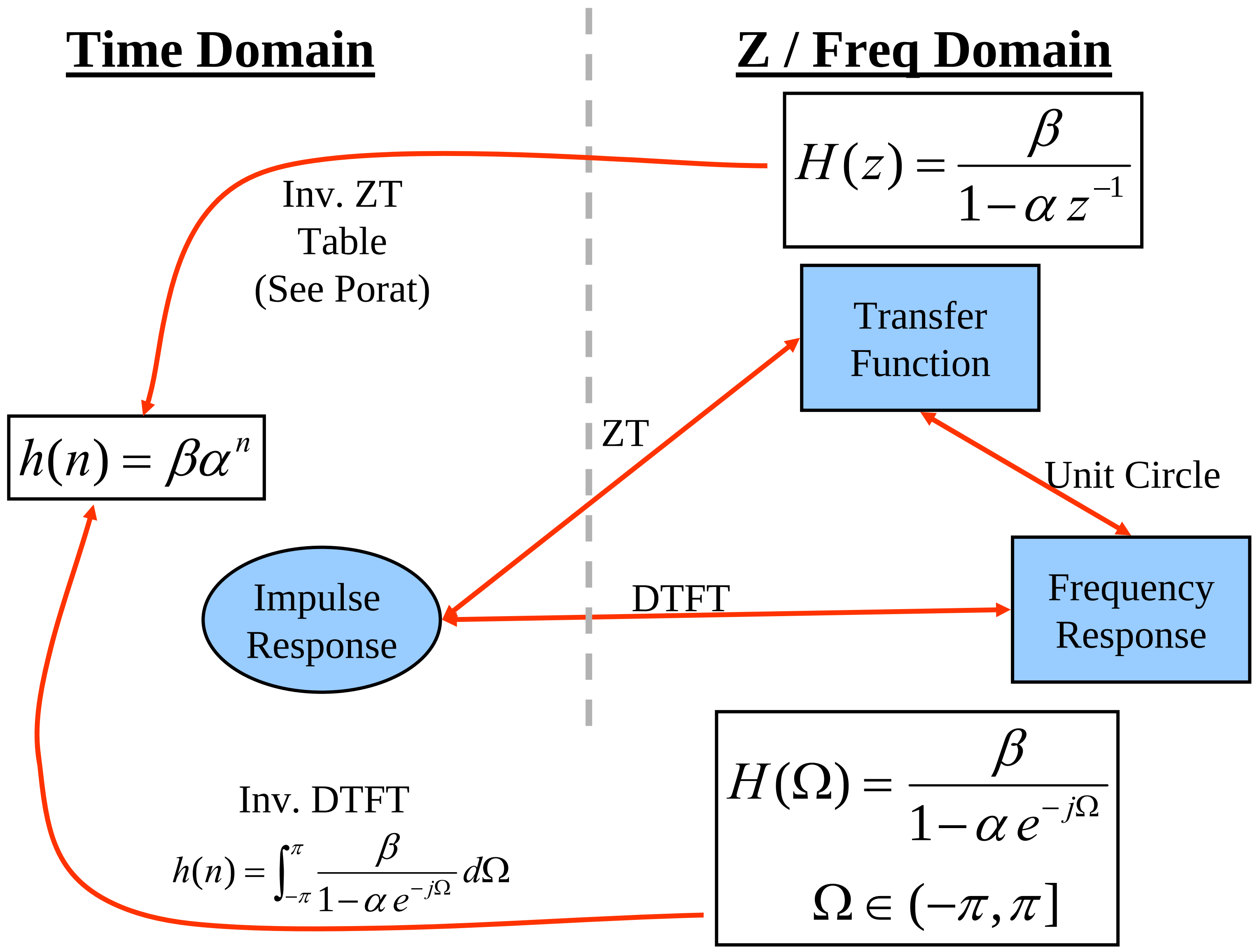
$$H(z) = \frac{\beta}{1-\alpha z^{-1}} = \frac{\beta z}{z-\alpha}$$

Zero
Num. = 0
When $z = 0$

Pole
Den. = 0
When $z = \alpha$

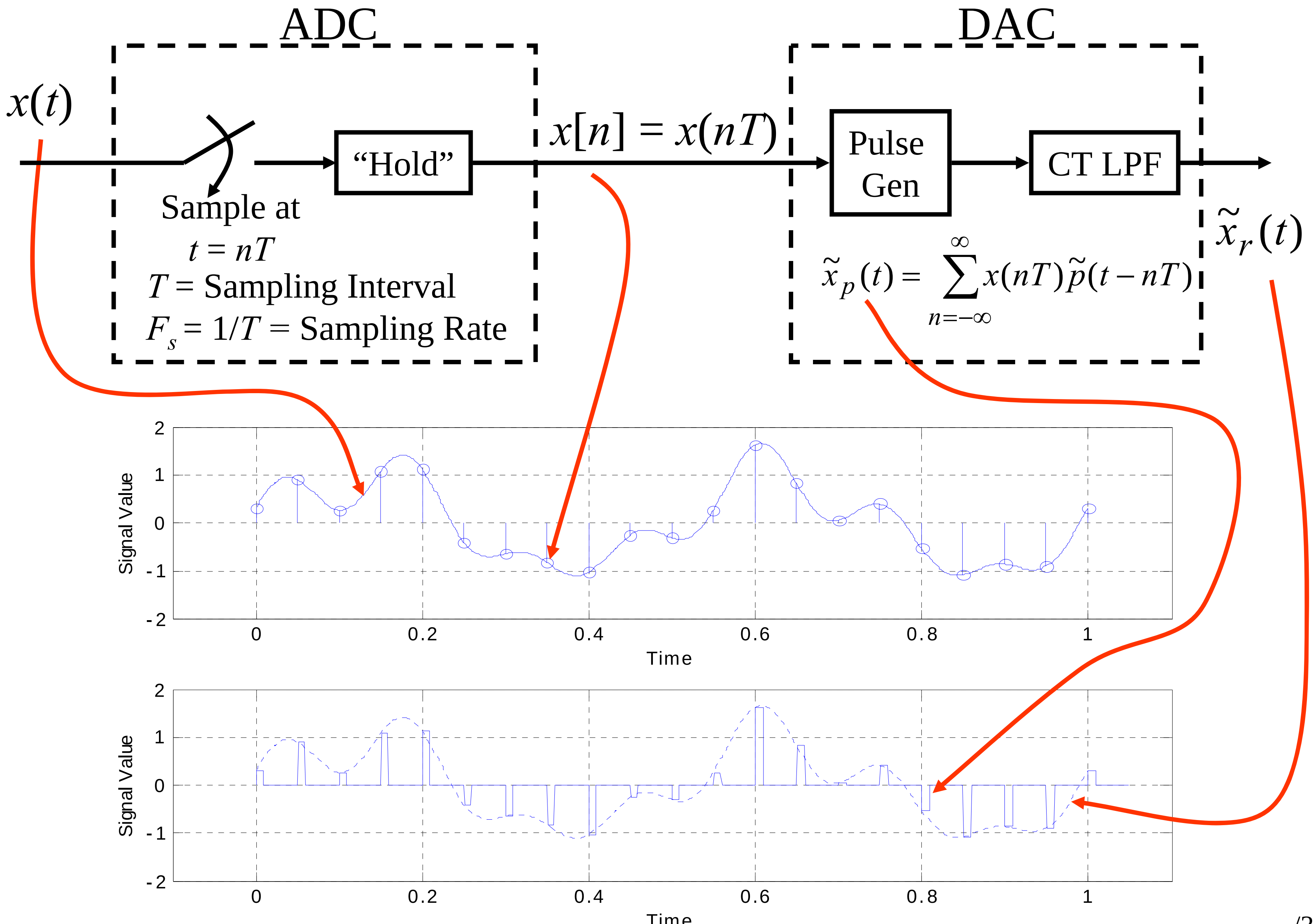
If $|\alpha| < 1$: **In**side UC
If $|\alpha| > 1$: **Out**side UC

Example System (cont.)



Review of Standard Sampling Theory

Practical Sampling Set-Up



Sampling Analysis (#1)

Goal = Determine Under What Conditions We Have:

Reconstructed CT Signal = Original CT Signal $\tilde{x}(t) = x(t)$

Simplify to Develop Theory: Use $\tilde{p}(t) = \delta(t)$

Why????

1. Because delta functions are EASY to analyze!!!
2. Because it leads to the best possible case

$$\boxed{\tilde{x}_p(t) \quad \Rightarrow \quad x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)}$$

$x_p(t)$ is called the "Impulse Sampled" signal

Note: $x_p(t)$ shows up during Reconstruction, not Sampling!!

Sampling Analysis (#2)

Now, the impulse sampled signal $x_p(t)$ is filtered to give the reconstructed signal $x_r(t)$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

$$= x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$\longrightarrow$ $\boxed{\text{CT LPF}}$ $\longrightarrow$

$H(f)$

$$x_r(t) = ???$$

$$\updownarrow$$

$$X_r(f) = H(f)X_p(f)$$

Need to find $X_p(f)$…. But First a Trick!!!

$$x_p(t) = x(t)\underbrace{\sum_{n=-\infty}^{\infty}\delta(t-nT)}_{\text{periodic} \Rightarrow \text{use FS}}$$

$$= x(t)\frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j2\pi kt/T}$$

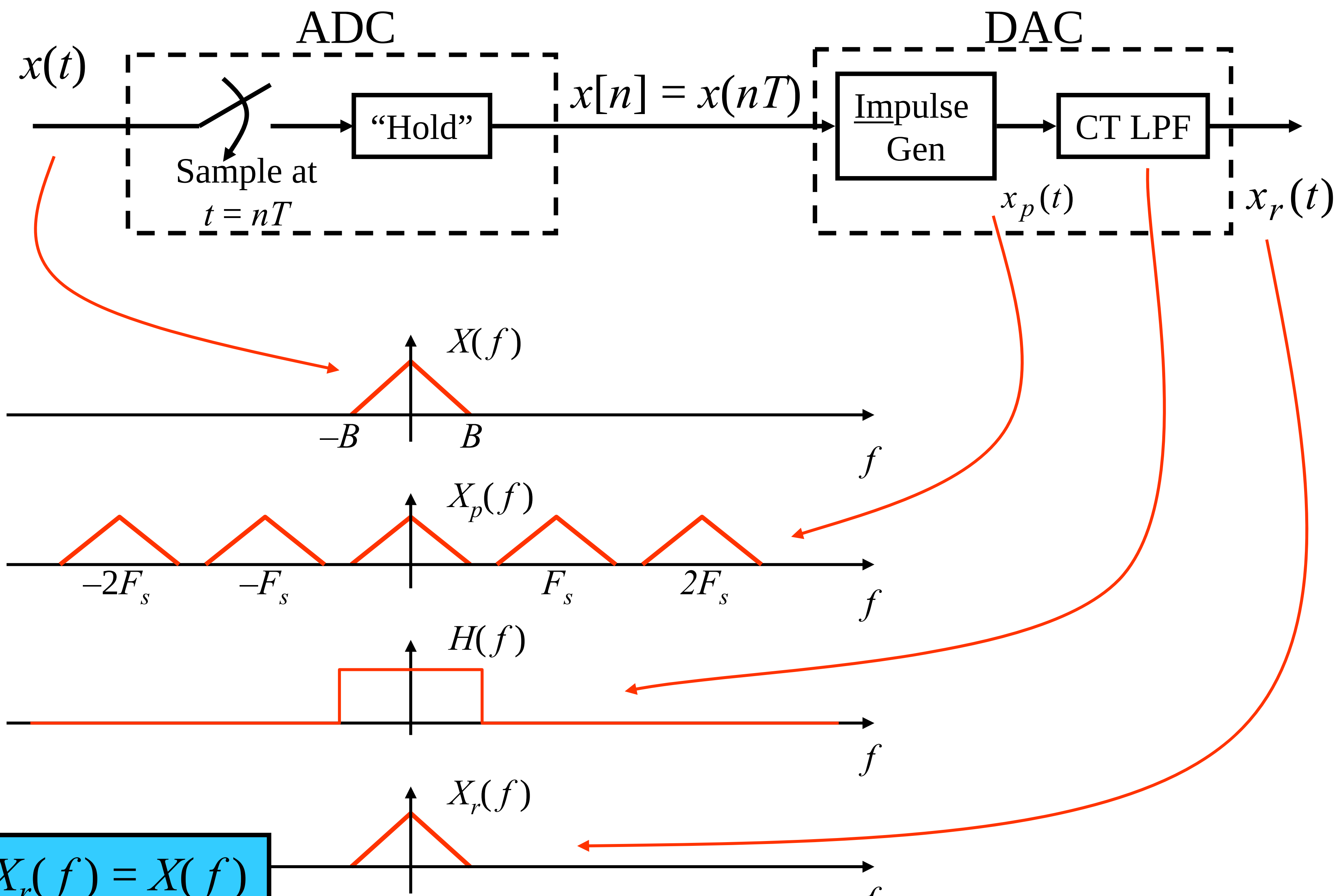
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} x(t)e^{j2\pi kF_s t}$$

$\Rightarrow$

$$X_p(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F\left\{x(t)e^{j2\pi kF_s t}\right\}$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f-kF_s)$$

Sampling Analysis (#3)



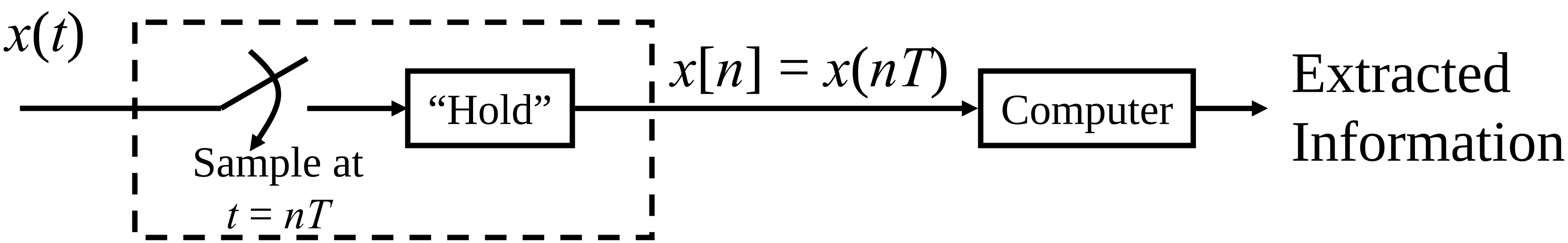
$X_r(f) = X(f)$
If $F_s \geq 2B$

Sampling Analysis (#4)

What this says: Samples of a bandlimited signal completely define it as long as they are taken at $F_s \geq 2B$

Impact: To extract the info from a bandlimited signal we only need to operate on its (properly taken) samples

➔ Use computer to process signals



Sampling Analysis (#5)

FT of Impulse Sampled Signal gives view of Original FT

&

FT of Impulse Sampled Signal = DTFT of Samples

➔ DTFT gives view of FT of original signal

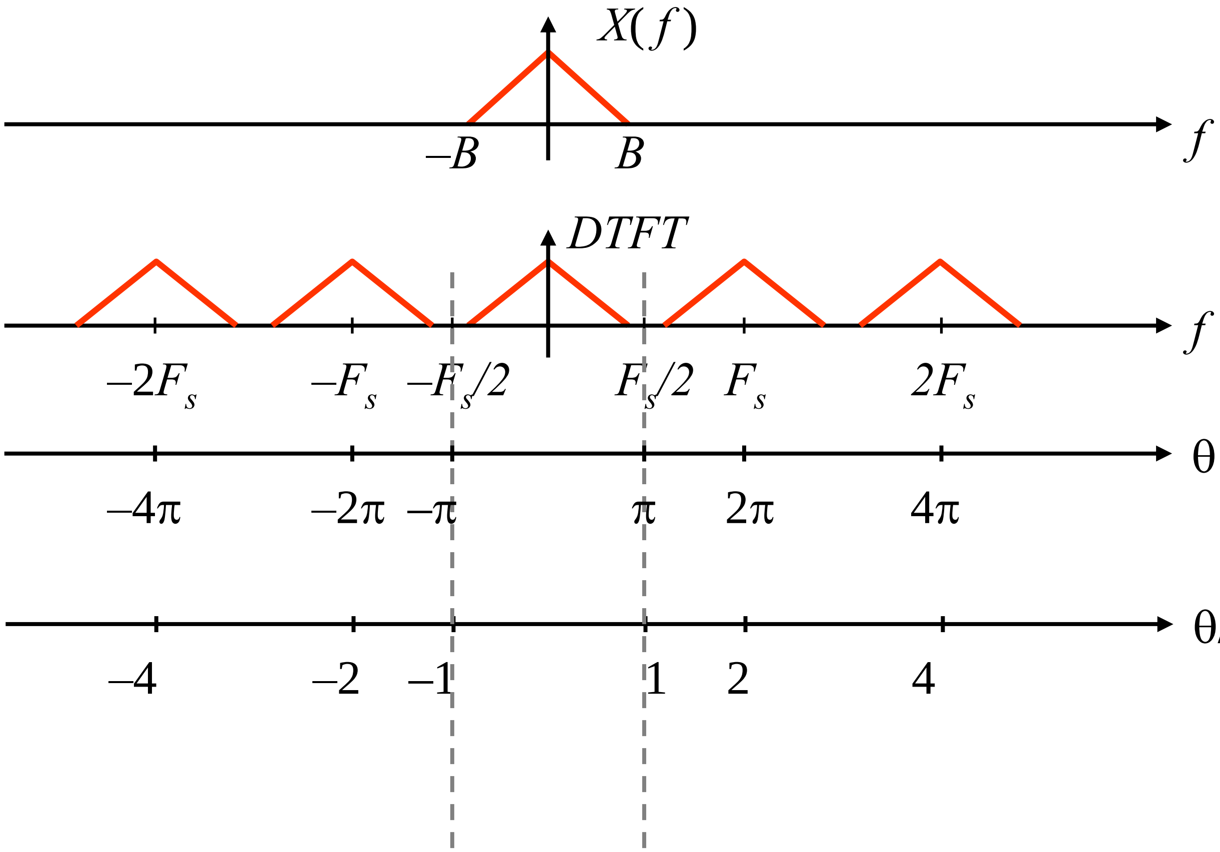
Let's see this

$$F\{x_p(t)\} = F\left\{\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)\right\} = \sum_{n=-\infty}^{\infty} x(nT)\underbrace{F\{\delta(t-nT)\}}_{e^{-jn\omega T}}$$

$$= \sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T} = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\theta} \qquad \theta = \omega T$$

$$\boxed{X_p^F(\theta / T) = X^f(\theta)}$$

Sampling Analysis (#6)



$$\theta = \omega T$$

$$= \left[\frac{2\pi}{F_s}\right] f$$

Read notes on web on "Concept of Digital Frequency"

Introduction to EE521 Case Study

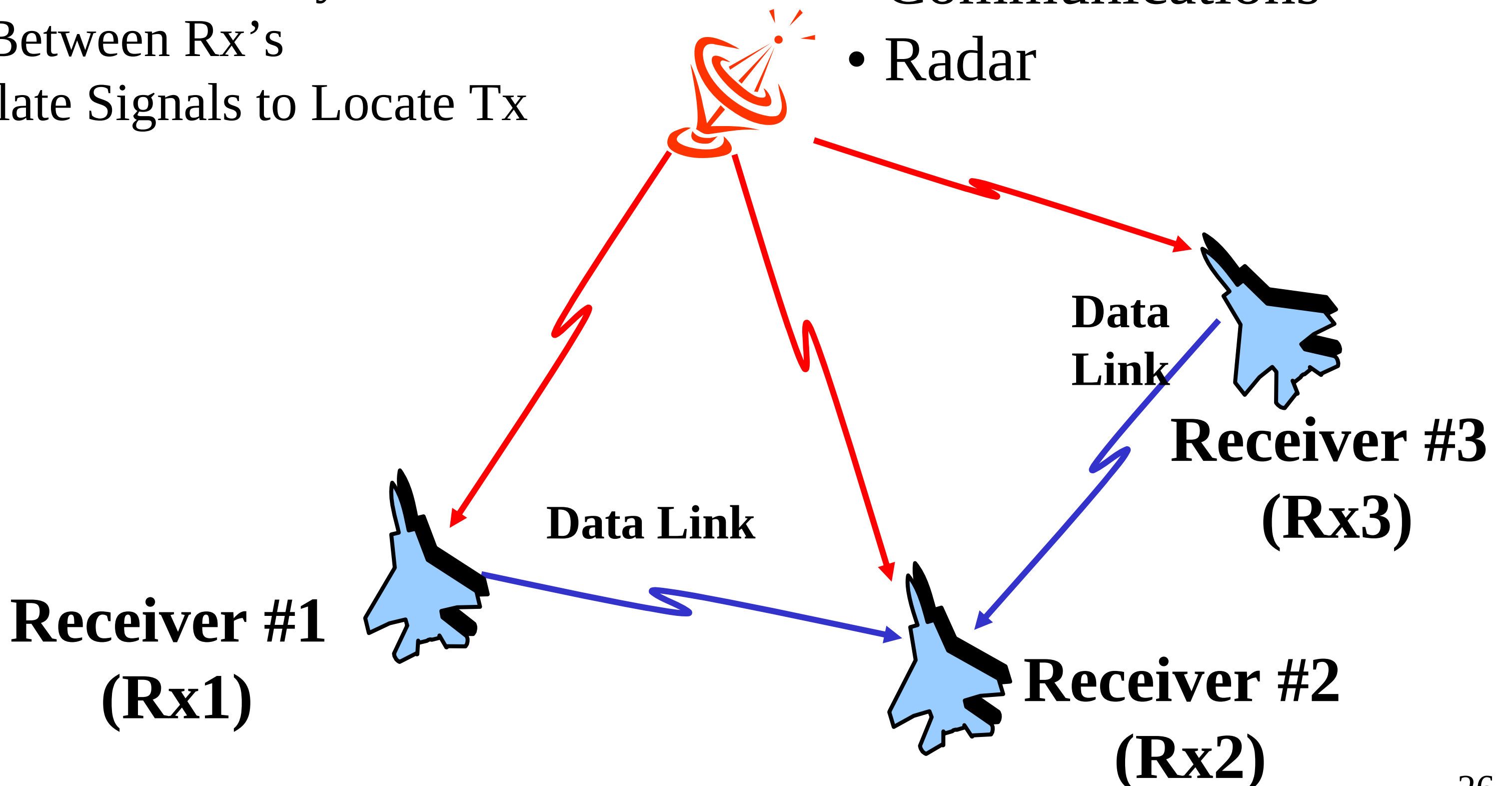
EE521 Case Study: Emitter Location

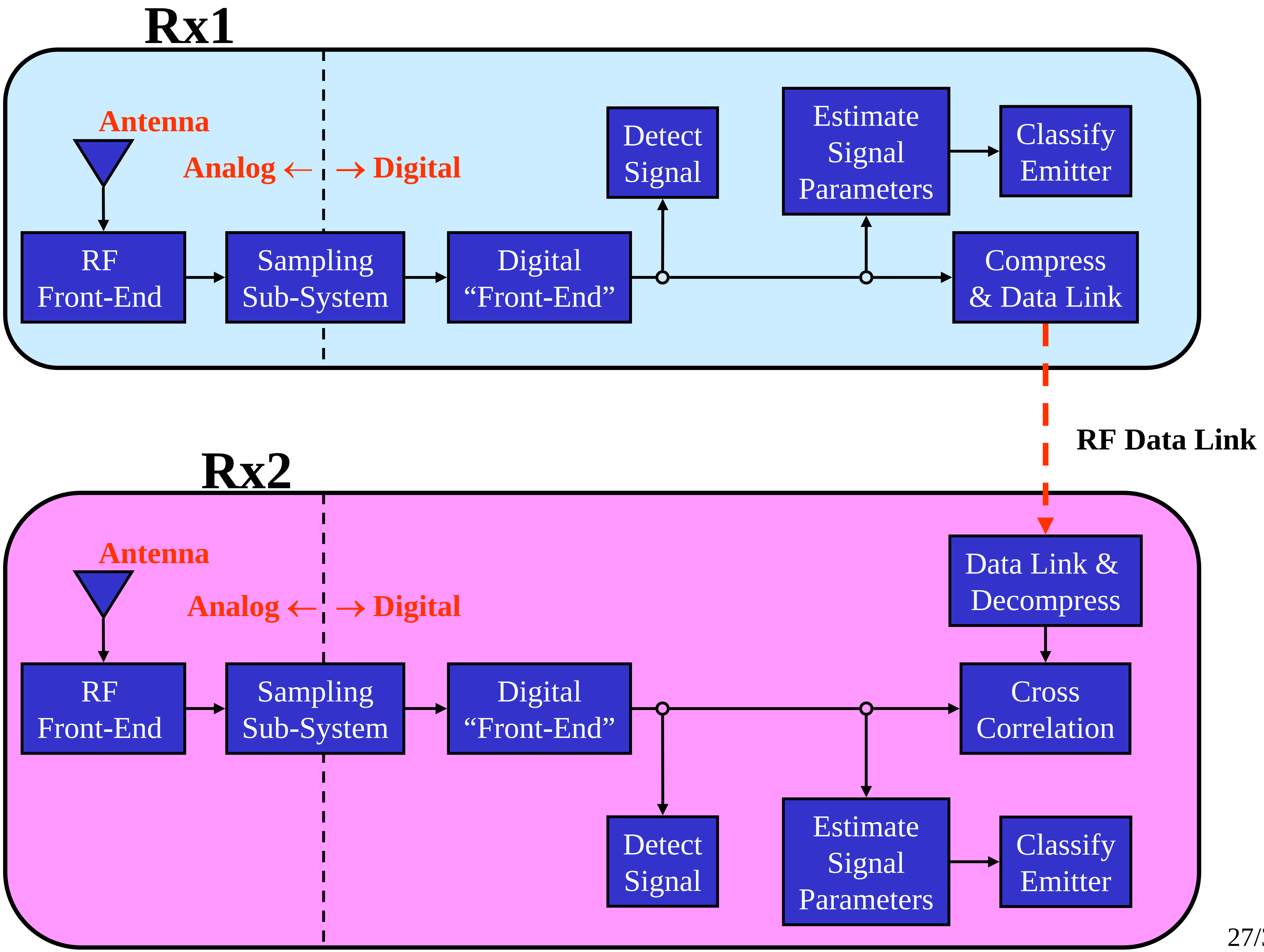
Processing Tasks

- Intercept RF Signal @ Rx’s
- Sample Signal Suitably for Processing
- Detect Presence of Emitter’s Signal
- Estimate Characteristics of Signal
- Use Est’d Char’s to Classify Emitter
- Share Data Between Rx’s
- Cross-Correlate Signals to Locate Tx

Radio Freq Transmitter (Tx)

- Communications
- Radar



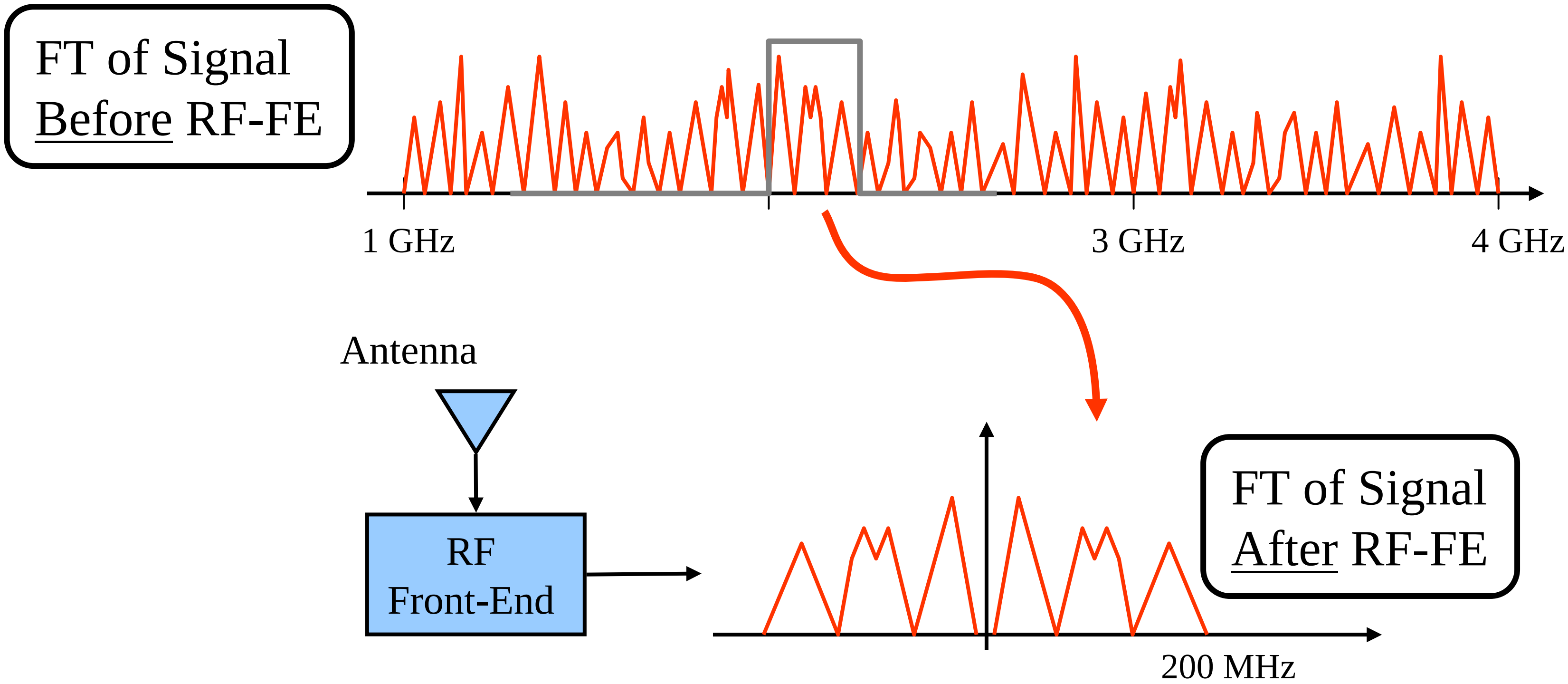

Rx1
Antenna
Analog ← → Digital
RF Front-End
Sampling Sub-System
Digital “Front-End”
Detect Signal
Estimate Signal Parameters
Classify Emitter
Compress & Data Link
RF Data Link
Rx2
Antenna
Analog ← → Digital
Data Link & Decompress
RF Front-End
Sampling Sub-System
Digital “Front-End”
Cross Correlation
Detect Signal
Estimate Signal Parameters
Classify Emitter

Processing Tasks Overview (#1)

RF Front-End (Analog Circuitry): “RF” = “Radio Frequency”

- Selects reception band; may need to scan bands
- Amplifies signal to level suitable for ADC, etc.
- Frequency-shifts signal spectrum to range suitable for ADC
  - Technology trend is toward requiring less shifting
- Unfortunately: noise introduced by analog electronics (***Random Signals***)

Topic We’ll Cover



Processing Tasks Overview (#2)

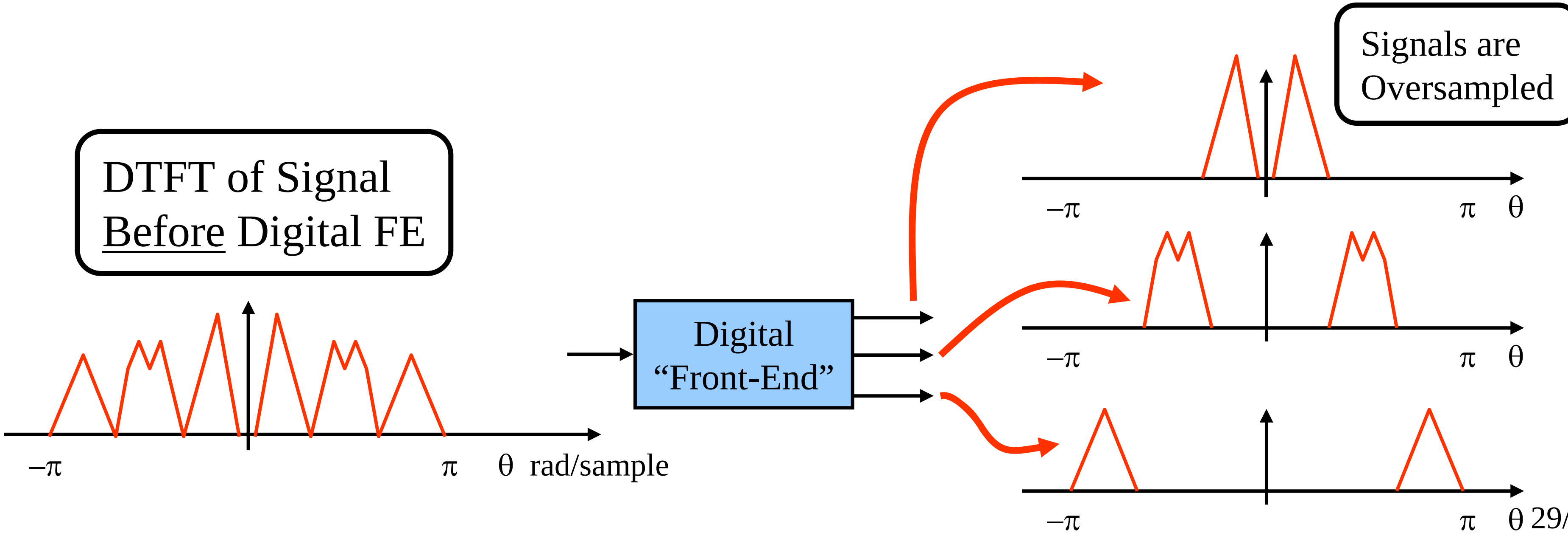
Sampling Sub-System (Mixed Analog/Digital Circuitry):

- Converts analog CT signal into digital DT signal (***Bandpass Sampling***)
- Converts real-valued RF signal into complex LP signal (***Equiv. LP Signals***)

Digital Front-End (Digital Processing):

- Convert to complex LP signal (***Equiv. LP Signals***)
- Equalizes response of analog front-end (***DFT-Based Filtering***)
    - Magnitude and Phase
- Bank of digital filters splits signal into sub-bands for further processing (***Filter Banks***)
- Reduce sampling rate after filtering to sub-bands (***Multi-Rate Processing***)

Topics We'll Cover



Processing Tasks Overview (#3)

Detect Presence of Signal (Digital Processing):

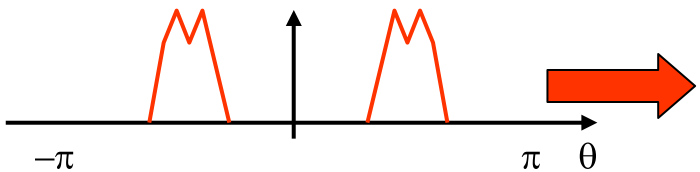
- Has signal been intercepted or only noise in subband (***DFT-Based Proc.***)

Estimate Parameters of Signal (Digital Processing) (see also EE522):

- Estimate frequency of signal (***DFT-Based Processing***)

Classify/Model Signal (Digital Processing):

- What type of signal is it? (***Spectral Analysis of Random Signals***)



PSD Model

$$P_x(\theta) = \sigma^2 \left[\frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_p z^{-p}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_q z^{-q}} \right]_{z = e^{j\theta}}$$

Use PSD Model parameters $\{b_i\}$ and $\{a_i\}$ to classify signal type

Compress/De-Compress Signal (Digital Processing) (see also EE523):

- Exploit signal structure to allow efficient transfer
  (***DFT-Based Proc./ Filter Banks / Spectral Analysis***)

Cross-Correlate Signals (Digital Processing):

- Compute relative delay and Doppler (***DFT-Based Proc.& Multi-Rate Proc.***)