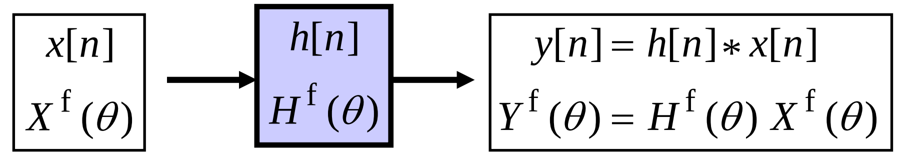
DFT-Based FIR Filtering

See Proakis & Manolakis 7.3

Motivation: DTFT View of Filtering

There are two views of filtering:

* Time Domain
* Frequency Domain



The FD viewpoint is indispensable for analysis and design of filters:

* Passband, Stopband, etc. of $|H^{\mathrm{f}}(\theta)|$
* Linearity of Phase $\angle H^{\mathrm{f}}(\theta)$, etc.

Q: What about using DTFT for implementation?

* Compute DTFT of input signal and filter
* Multiply the two and take inverse DTFT

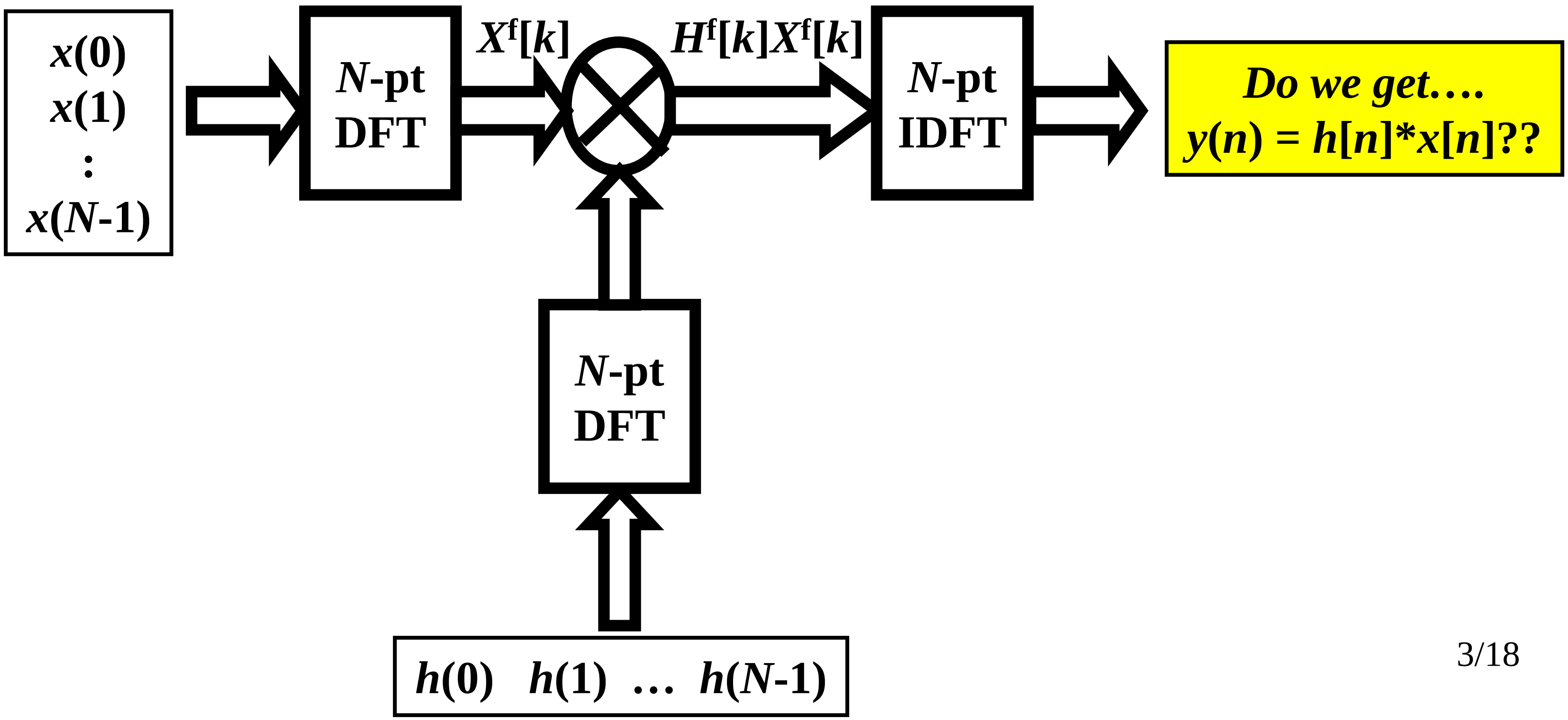
A: NO!!! Can't compute DTFT – must compute at infinite many frequency values

Desired Intention: But Does It Work?

But wait….

- If input signal is finite length, the DFT computes "samples of the DTFT"
- Likewise, if filter impulse response is finite length

Q: So… can we use this?



DFT Theory and Cyclic Convolution

A: Not Necessarily!!!!

DFT Theory (Sect. 7.2.2 in Proakis & Manolakis) tells us:

$$\mathbf{IDFT}\{H^{\mathrm{f}}[k]X^{\mathrm{f}}[k]\} = h[n] \circledast x[n]$$

Circular (Cyclic) Convolution

Thus… this block diagram gives something called cyclic convolution, not the "ordinary" convolution we want!!!

Q: When *does* it work???

A: Only when we "trick" the DFT Theory into making *circular* = *linear* convolution!!!!!!

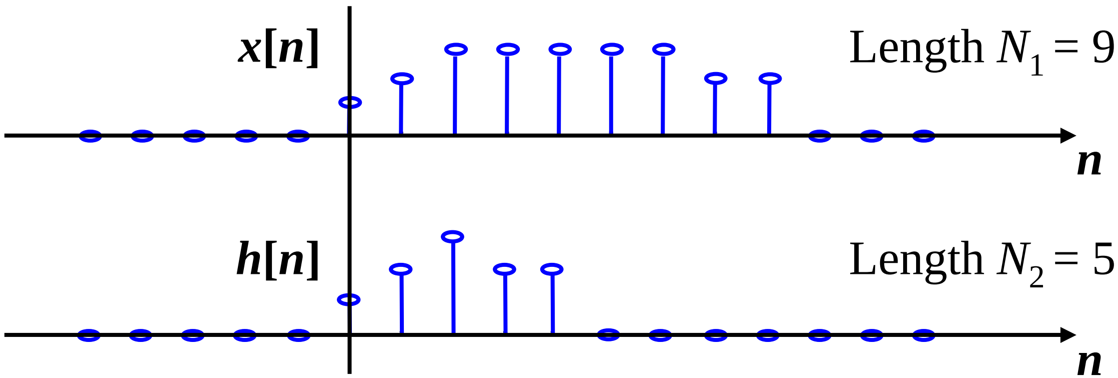
Q: So… when does Cyclic = Linear Convolution???

Easiest to see from an example!!!!!

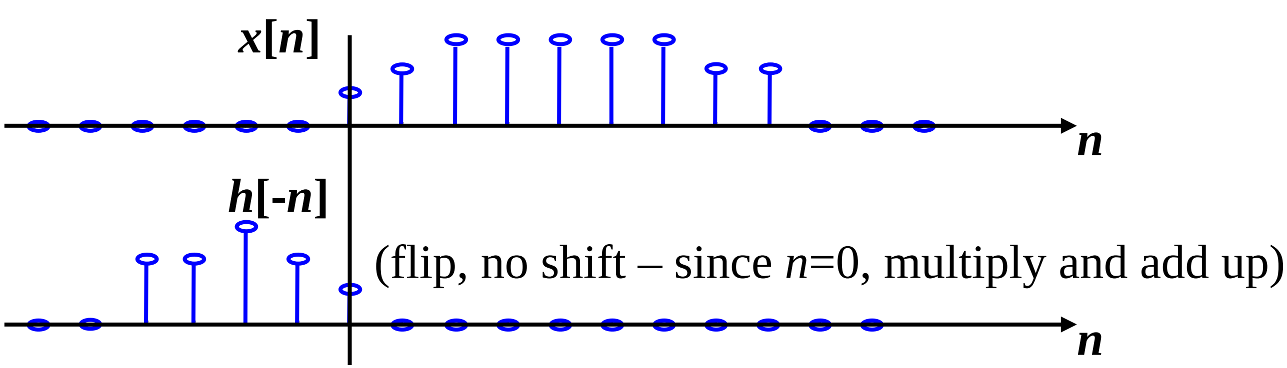
Linear Convolution for the Example

What does ***linear* convolution** give for 2 finite duration signals:

Original Signals:

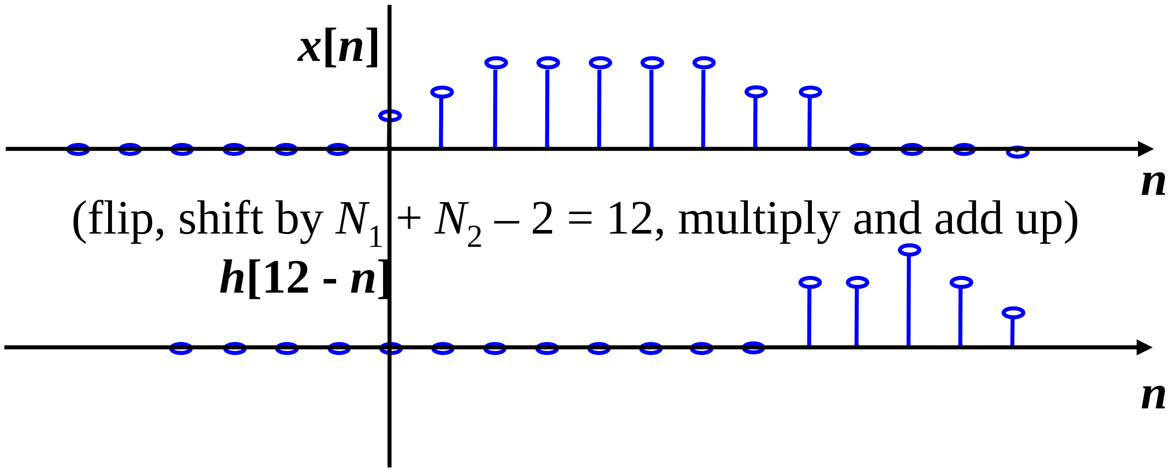


First Non-Zero Output is at n=0:



Linear Convolution for the Example (cont.)

Last Non-Zero Output is at $n = N_1 + N_2 - 2 = 12$:



The non-zero outputs are for $n = 0, 1, \ldots, 12$ ➔ 13 of them

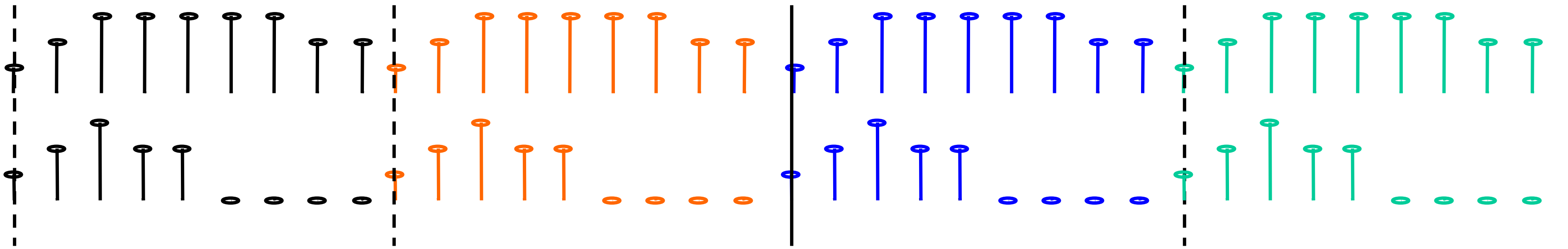
In General: Length of Output of Linear Convolution = $N_1 + N_2 - 1$

Cyclic Convolution for the Example

Now… What does ***cyclic convolution*** give for these 2 signals:

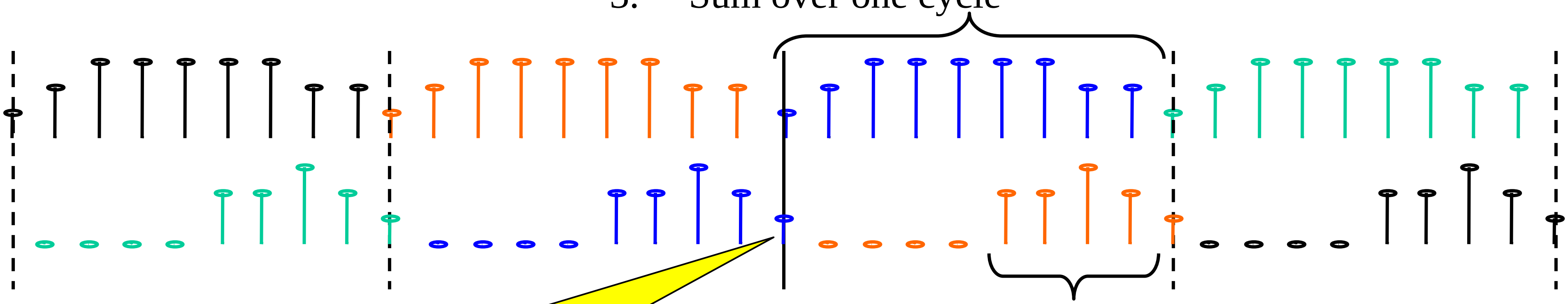
“Original” Signals:

1. Zero-Pad Shorter Signal to Length of Longer One
2. Then Periodize Each



“First” Output Sample:

1. Flip periodized version around this point
2. No shift needed to get $n = 0$ Output Value
3. Sum over one cycle



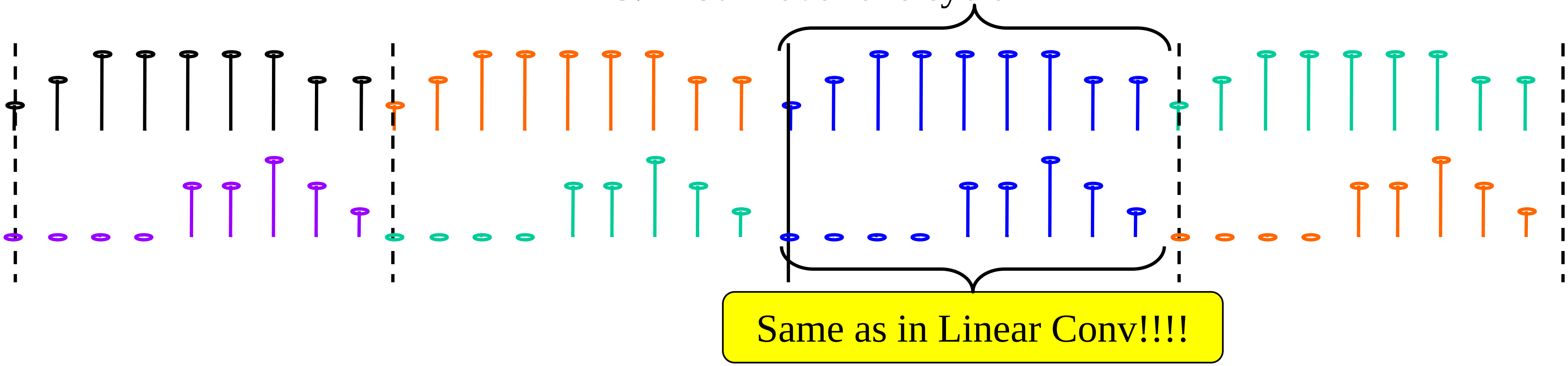
Same as in Linear Conv!!!!

Not Present in Linear Conv!!!!

Linear Convolution for the Example (cont.)

“Last” Output Sample (i.e., n = 8):

1. Flip periodized version around this point
2. Shift by 8 to get n = 8 Output Value
3. Sum over one cycle



Note: If I try to compute the output for $n = 9$ ➔ Exactly the same case as for $n = 0$!!
Thus, the output is cyclic (i.e., periodic) with unique values for n = 0, 1, …. 8

In General: Length of Output of Cyclic Convolution = $\max\{N_1, N_2\}$

<u>Making Cyclic = Linear Convolution</u>

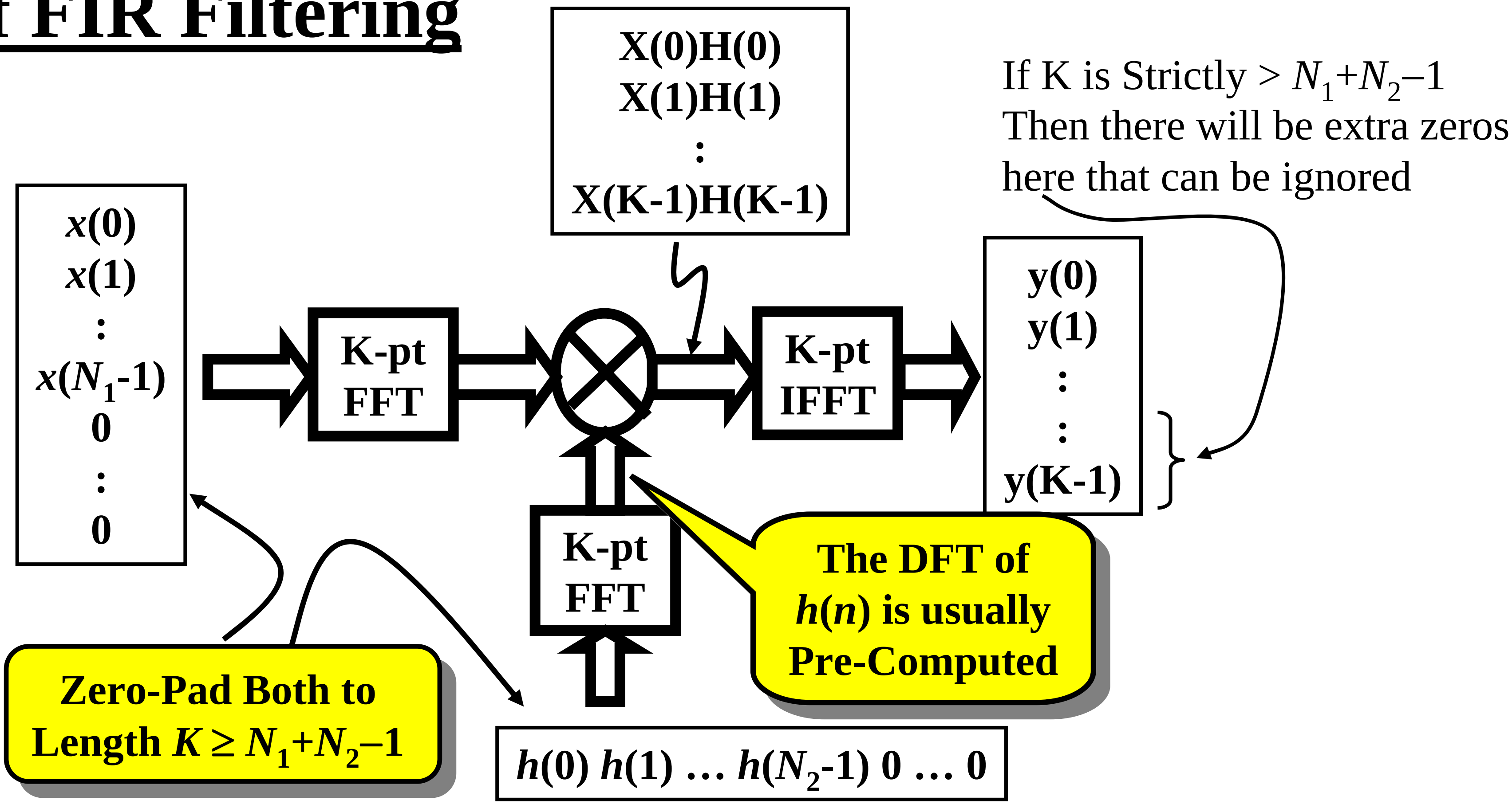
So…. Some of the output values of cyclic conv are <u>different</u> from linear conv!!!
Some of the output values of cyclic conv are <u>same as</u> linear conv
And….
The length of cyclic conv differs from the length of linear conv!!!

From the example above we can verify:
If we choose $K \geq N_1 + N_2 - 1$
And Zero-Pad Each Signal to Length K
Then <u>Cyclic = Linear Convolution</u>

N_1 = Length of $x[n]$
N_2 = Length of $h[n]$

<u>Exercise</u>: Verify this for the above example!!!!

Simple Frequency-Domain Implementation of FIR Filtering



Why Do This?

The FFT's Efficiency Makes This Faster Than Time-Domain Implementation (In Many Cases)

Problems with the *Simple* FD Implementation

Q: What if $N_1 >> N_2$?
A: Then, need **Really Big FFT** ➔ Not Good!!!
(Input signal much longer than filter length)
Also… can't get any output samples until after whole signal is available and FFT processing is done. **Long Delay**.

Example: Filter 0.2 sec of a radar signal sampled at Fs = 50 MHz
$N_1 = (0.2 \text{ sec})\times(50\times10^6 \text{ samples/sec}) = 10^7$ samples
FFT Size > 10^7 ➔ Really Big FFT!!!!

Q: What if N_1 is unknown in advance?
Example: Filtering a stream of audio
A: FFT size can't be set ahead of time – difficult programming

***Simple* FD Implementation Has Serious Limitations!!!**

Better **FD-Based FIR Filter Implementations**

Two Very Similar Methods Exist

- Overlap – Add (OLA) ← Covered Here
- Overlap – Save (OLS)

<u>Both</u> methods exploit linearity of filter:

- Break input signal into a sum of blocks → $x[n] = \sum_i x_i[n]$
- Output = sum of response to each block

$$
\begin{aligned}
z[n] &= (x * h)[n] \\
&= \sum_i \underbrace{(x_i * h)[n]}_{=z_i[n]} \\
&= \sum_i z_i[n]
\end{aligned}
$$

Use the *Simple* FD-Based Method to Compute Each Output Block

<u>The difference between OLA & OLS</u>

lies in how the $x_i[n]$ blocks are formed

OLA Method for FD-Based FIR

For OLA: Choose $x_i[n]$ to be non-overlapped blocks of length N_B

(blocks are contiguous)

N_B is a Design Choice

$$x_i[n] = \begin{cases} x[n], & iN_B \le n < (i+1)N_B \\ 0, & \text{otherwise} \end{cases}$$

<See Fig. 5.8 (a) on next slide>

Q: Now what happens when each of these length-N_B blocks gets convolved with the length-N_2 filter?

A: The output block has length $N_2 + N_B - 1 > N_B$

* Output Blocks are Bigger than Input Blocks
* But are separated by N_B points
* Thus… Output Blocks Overlap
* Total Output = "Sum of Overlapped Blocks"

"Overlap-Add"

<See Fig. 5.8 (b) – (d) on next slide>

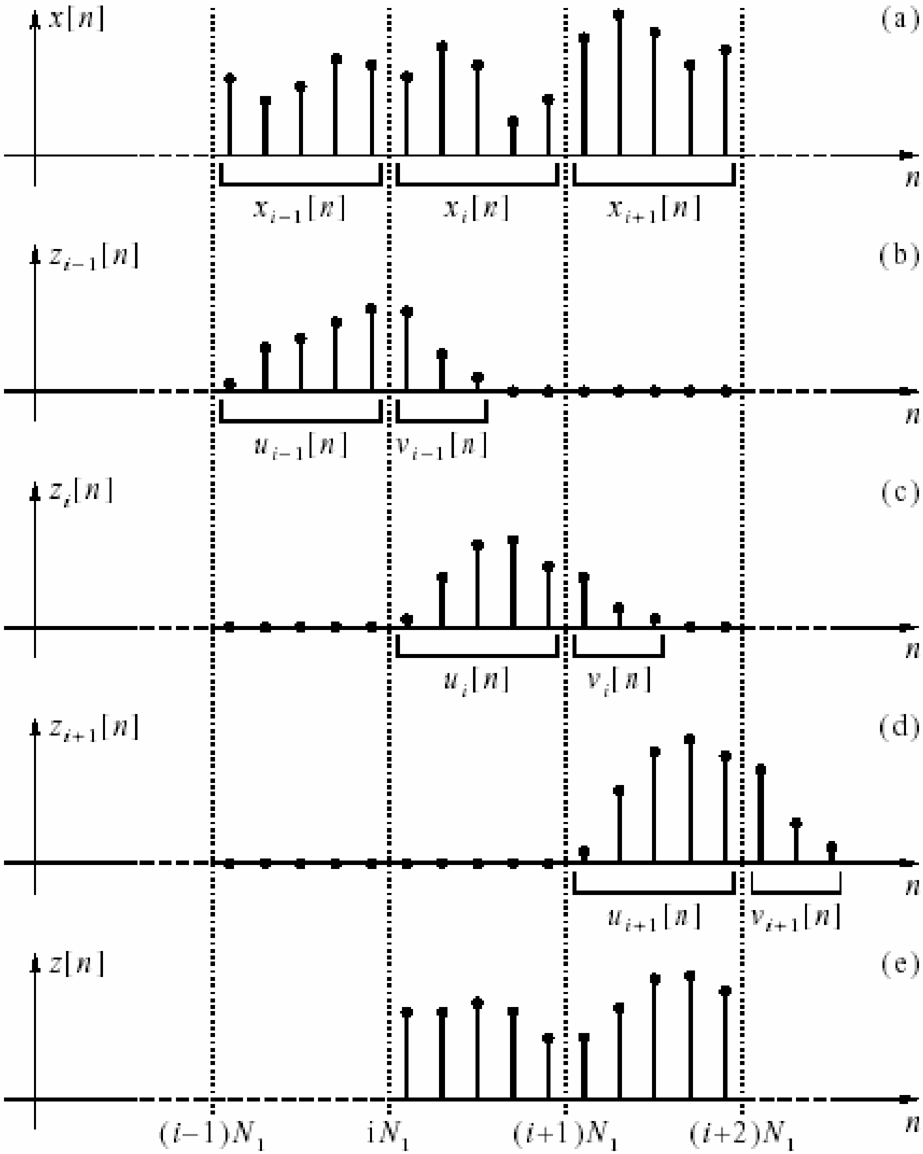


Figure from Porat's Book

Figure 5.8 Illustration of overlap-add convolution: (a) the input signal $x[n]$; (b), (c), (d) the three output segments $z_{i-1}[n]$ through $z_{i+1}[n]$; (e) a segment of the output signal $z[n]$ corresponding to $x_i[n]$ and $x_{i+1}[n]$.

OLA Method Steps

Assume: Filter $h[n]$ length N_2 is specified

Choose: Block Size N_B & FFT Size $N_{\text{FFT}} = 2^p \geq N_B + N_2 - 1$

Choose N_B such that:

$N_B + N_2 - 1 = 2^p$

It gives minimal complexity for method (see below)

Run:

Zero-Pad $h[n]$ & Compute N_{FFT}-pt FFT (can be pre-computed)

For each i value ("For Each Block")

- Compute $z_i[n]$ using Simple FD-Based Method
  - Zero-Pad $x_i[n]$ & Compute N_{FFT}-pt FFT
  - Multiply by FFT of $h[n]$
  - Compute IFFT to get $z_i[n]$
- Overlap the $z_i[n]$ with previously computed output blocks
- Add it to the output buffer

<See Fig. 5.8 (e) on previous slide>

OLA Method Complexity

- The FFT of filter $h[n]$ can be pre-computed ➔ Don't Count it!
- We'll measure complexity using # Multiplies/Input Sample
- Use $2N_{FFT}\log_2 N_{FFT}$ Real Multiplies as measure for FFT
- Assume input samples are Real Valued

  Can do 2 real-signal FFT's for price of **≈ 1 Complex FFT** (Classic FFT Result!)

- For Each Pair of Input Blocks
  - One FFT: $2N_{FFT}\log_2 N_{FFT}$ Real Multiplies
  - Multiply DFT × DFT: $4N_{FFT}$ Real Multiplies
  - One IFFT: $2N_{FFT}\log_2 N_{FFT}$ Real Multiplies
  - Total: $4N_{FFT}[1 + \log_2 N_{FFT}]$ Real Multiplies

    $= 4(N_B + N_2 - 1)\,[1 + \log_2(N_B + N_2 - 1)]$
- The Number of Input Samples = 2 Blocks = $2N_B$

- **# Multiplies/Input Sample** $= 2(1 + (N_2 - 1)/N_B)\,[1 + \log_2(N_B + N_2 - 1)]$

Comparison to TD Method Complexity

Complexity of TD Method

- Filter $h[n]$ has length of N_2
- To get each output sample:
  - Multiply each filter coefficient by a signal sample: **N_2 Multiplies**
- **# Multiplies/Input Sample = N_2 Multiplies**

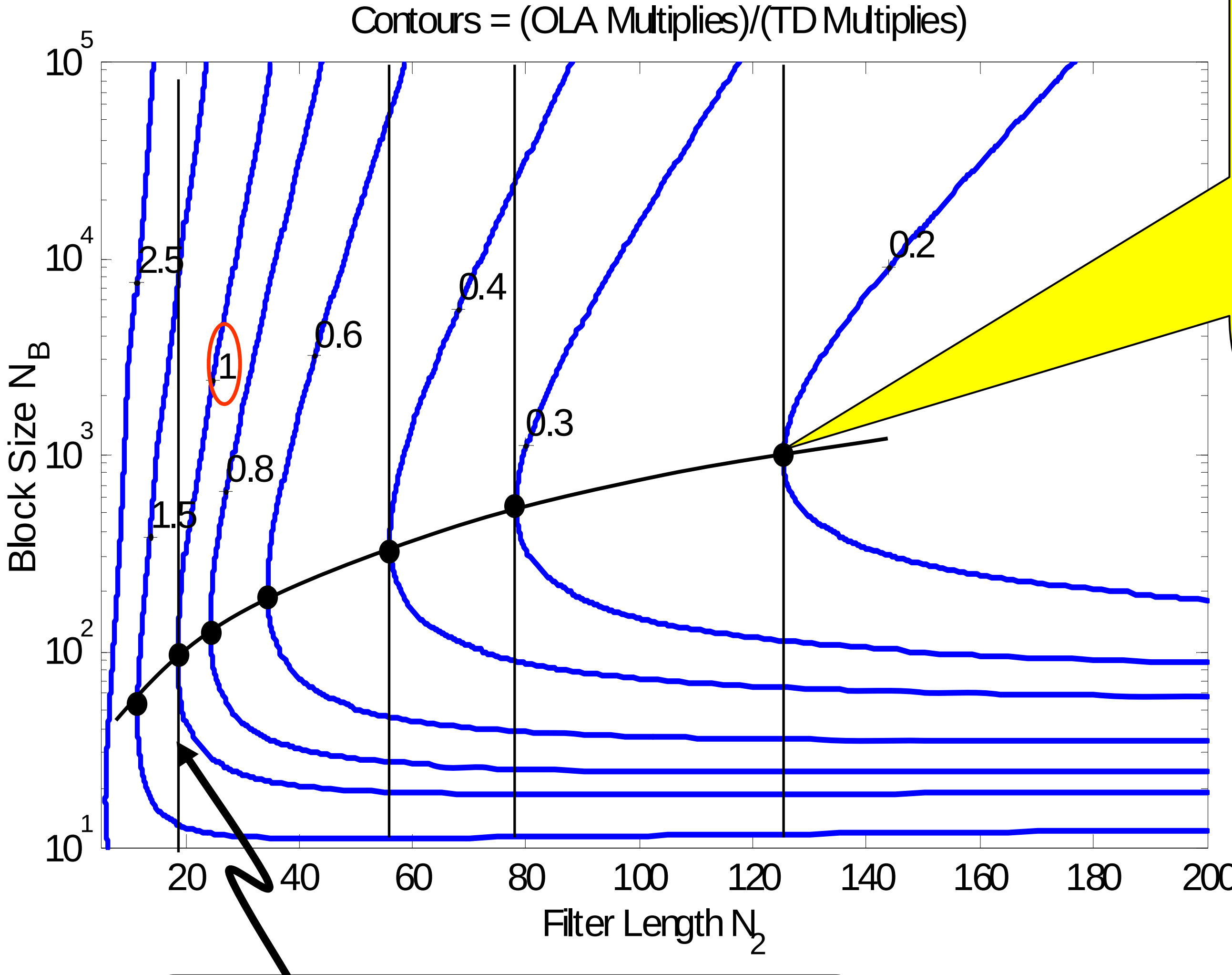
Condition Needed For OLA to Be More Efficient:

$$2\left(1+\frac{N_2-1}{N_B}\right)\left[1+\log_2\left(N_2+N_B-1\right)\right] < N_2$$

Thus… For a given N_2, Choose N_B to minimize the left-hand side

FD Complexity vs TD Complexity

Plot of: [Left-Hand Side]/[Right-Hand Side] of (5.38)



Optimal Block Size For Filter Length of ≈ 125 (Compare to Table 5.2 in Porat)

OLA More Efficient Only For Filters Longer than 19