Practical Spectral Analysis

(Porat Chapter 6)

<u>Goal of Practical Spectral Analysis</u>

<u>Goal</u>: Given a discrete-time signal $x[n]$, use DFT (via FFT) to analyze its spectral content – in particular, to detect the presence of sinusoids and estimate their frequency.

<u>Challenges</u>:

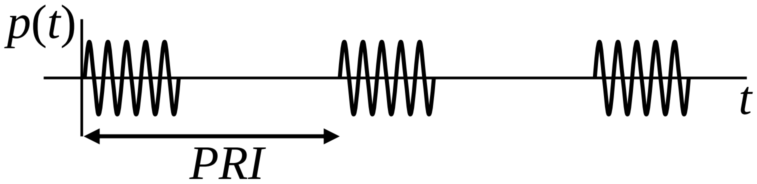
1. Available signal may be short (e.g., a radar signal may only be "on" for a very short time).
2. If the signal <u>is</u> long, then the spectral content may change with time (e.g., music spectrum changes with time) – so spectrum may be considered to be constant only a block-by-block basis where the blocks are short.

Both of these drive the need to apply the DFT to a short signal record ➔ **Challenge = Resolution & Accuracy**

Example Application (Electronic Warfare)

Intercept T seconds of a Radar Pulse Train $p(t)$, Compute DFT, detect & estimate peaks to identify type of radar.

"Underlying" Pulse Train is Periodic ➔ Fourier Series

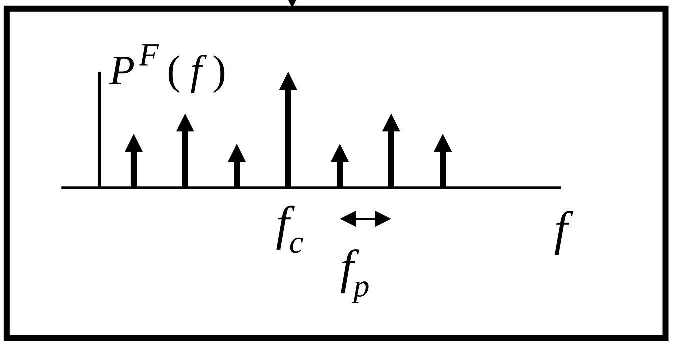


FS

$$p(t) = \sum_n c_n e^{jn2\pi(f_c - f_p)t}$$

$f_p = 1/PRI$

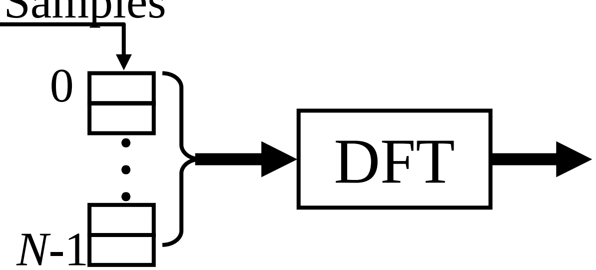
FT

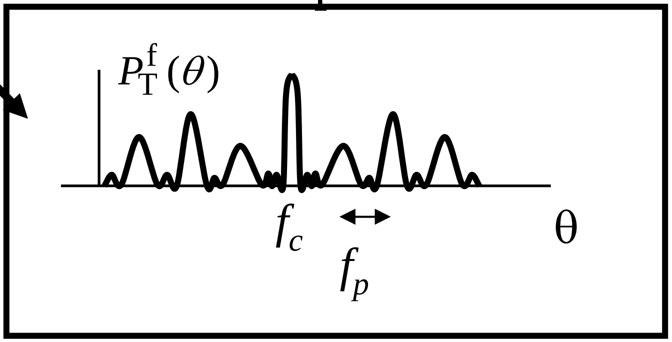


DTFT of **Infinite** Duration Signal

Since DFT shows samples of the DTFT of the finite duration signal we can study what the DFT gives us by looking at what the DTFT of a **finite-duration** signal looks like!!

DFT Shows Samples of This DTFT





DTFT of **Finite** Duration Signal

Effect of Windowing

Porat Sections 6.1 and 6.2

Basic Viewpoint of Signal Data

We are given a finite # of signal samples, and want to use them to see the spectrum of the infinite-duration signal….
How well can we do that?

Math Model for having a finite # of samples:

$$x[n] = \begin{cases} y[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Finite Duration Signal Available for Processing ($x[n]$)

Infinite Duration Signal ($y[n]$)

Better Math Model – Rectangular Window-Based Model:

$$x[n] = y[n] w_r[n], \quad \text{where } w_r[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Implication of Window-Based Model

Since the available data $x[n]$ is related to the unavailable signal $y[n]$ through multiplication we can use the Multiplication Theorem for DTFT (Eq. (2.103) in Porat) to find out what we get!

Thus, the DTFT of the signal data is related to the DTFT of the infinite-duration signal by:

$$X^{\mathrm{f}}(\theta) = \frac{1}{2\pi}\left\{Y^{\mathrm{f}} \circledast W_r^{\mathrm{f}}\right\}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y^{\mathrm{f}}(\lambda) W_r^{\mathrm{f}}(\lambda-\theta)\,d\lambda$$

where the DTFT of the rect. window is:

$$W_r^{\mathrm{f}}(\theta) = \sum_{n=0}^{N-1} 1 e^{-j\theta n} = \frac{1-e^{-j\theta N}}{1-e^{-j\theta}}$$

Use Geometric Sum

$$= \frac{e^{-j\theta N/2}\left[e^{j\theta N/2} - e^{-j\theta N/2}\right]/2j}{e^{-j\theta/2}\left[e^{j\theta/2} - e^{-j\theta/2}\right]/2j}$$

Use Euler!

$$= e^{-j\theta\frac{(N-1)}{2}} \frac{\sin(\theta N/2)}{\sin(\theta/2)}$$

$$D(\theta, N) \triangleq \frac{\sin(\theta N/2)}{\sin(\theta/2)}$$

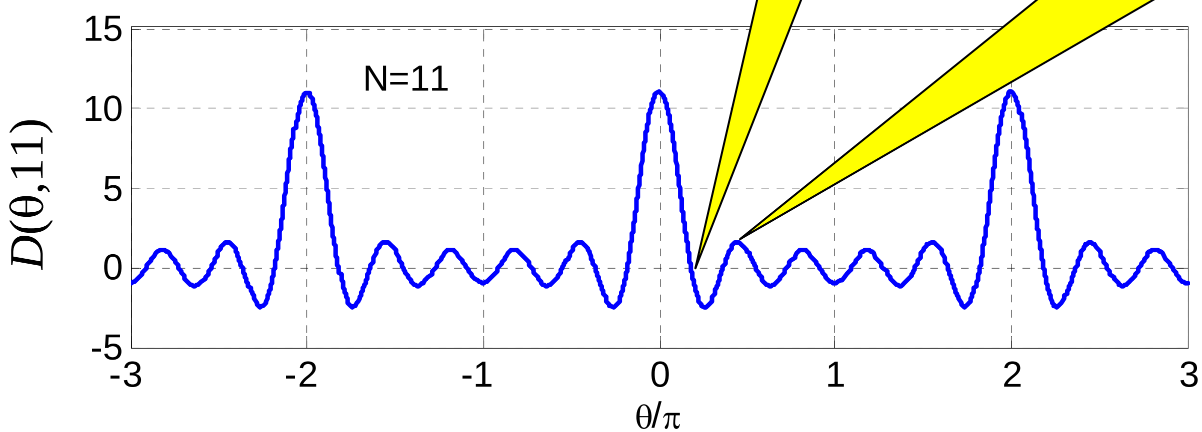
The "Dirichlet Kernel" $D(\theta,N)$

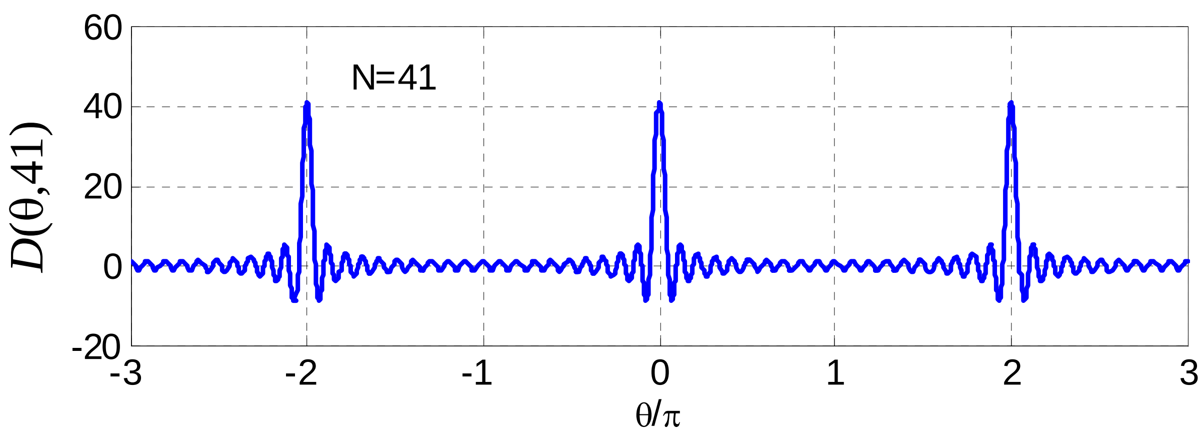
- Looks like "sinc", except periodic
- Mainlobe Gets Narrower as $N\uparrow$
- Sidelobes "Get Lower" as $N\uparrow$
- Height of Mainlobe = N

➔ **Looks more like delta as $N\uparrow$**

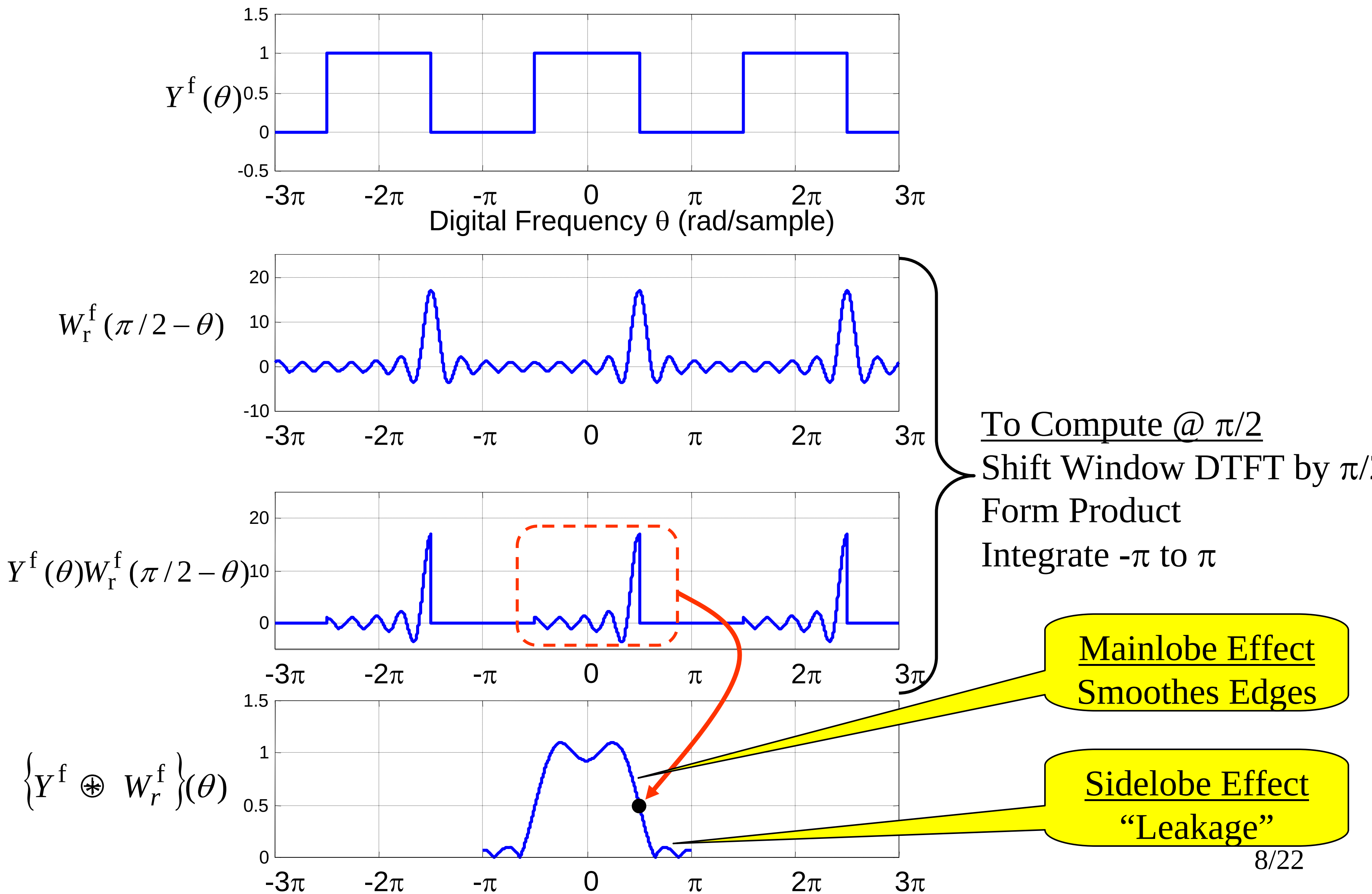
Nearest Zero @ $\theta=2\pi/N$ → Mainlobe Width = $4\pi/N$

Largest Sidelobe
–13 dB w.r.t. ML peak
(For Any N)





Impact of Window



Impact of Window (pt. 2)

Consider a signal consisting of two complex sinusoids:

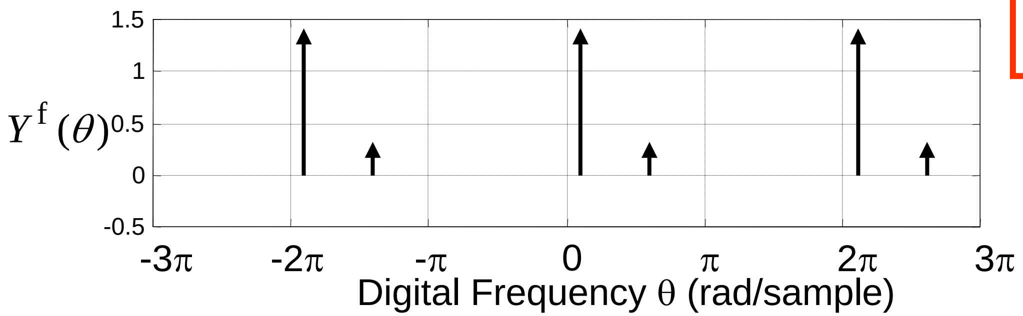
$$y[n] = A_1 e^{j\theta_1 n} + A_2 e^{j\theta_2 n}$$

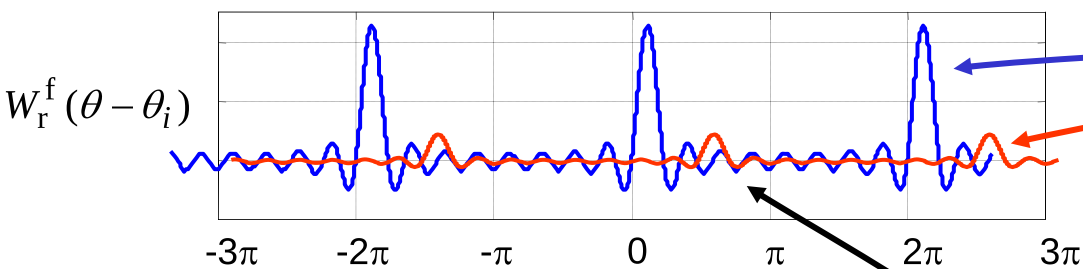
$$Y^{\mathrm{f}}(\theta) = A_1\delta(\theta-\theta_1) + A_2\delta(\theta-\theta_2), \quad \theta \in [-\pi, \pi] \text{ Repeats Elsewhere}$$

Recall: $F(\theta)*\delta(\theta-\alpha) = F(\theta-\alpha)$ so….

$$X^{\mathrm{f}}(\theta) = \frac{1}{2\pi}\left\{Y^{\mathrm{f}} \circledast W_r^{\mathrm{f}}\right\}(\theta)$$

$$= \frac{1}{2\pi}\left[A_1 W_r^{\mathrm{f}}(\theta-\theta_1) + A_2 W_r^{\mathrm{f}}(\theta-\theta_2)\right]$$



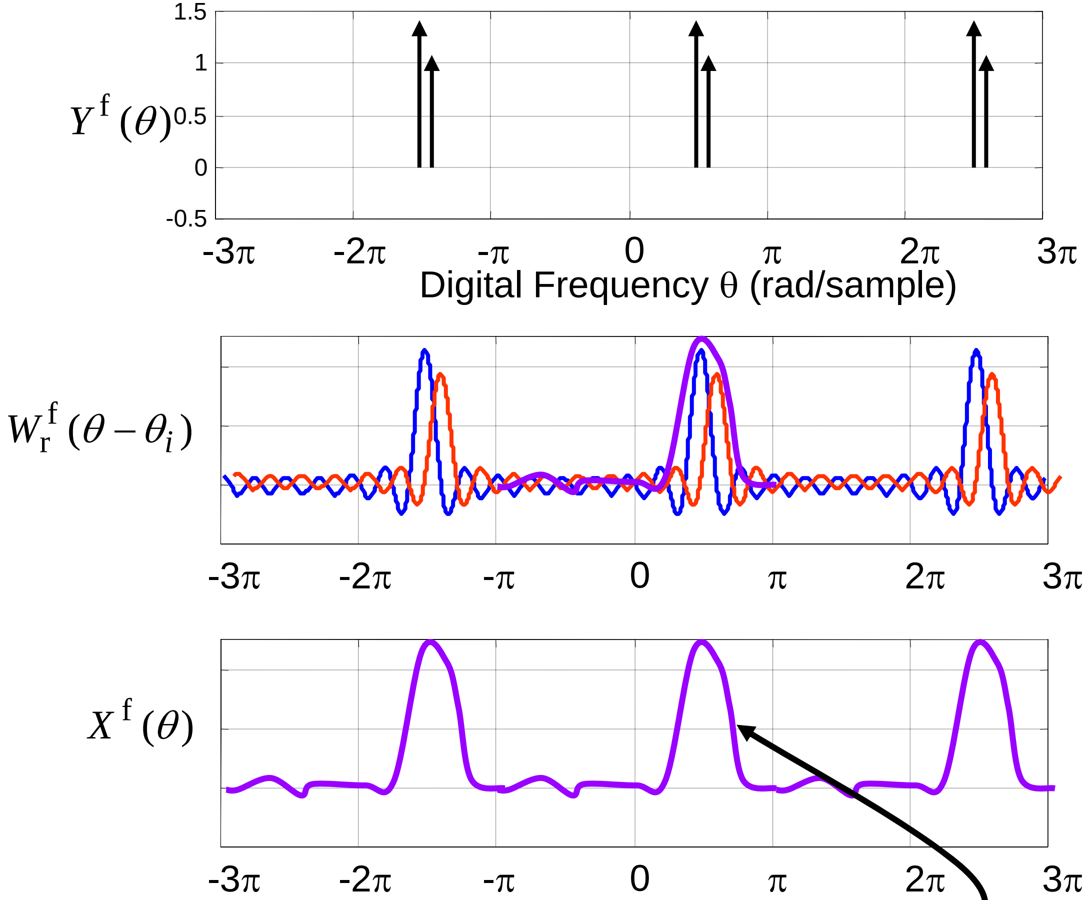


Sidelobe Leakage ("SL Interference")

Large Sidelobes Obscure Small Sinusoid

Impact of Window (pt. 3)

Consider a signal consisting of two complex sinusoids **closely spaced** in frequency and similar in amplitude:



Mainlobe Smearing

Wide Mainlobe "Smears" Sinusoids Together

Common Windows

Porat Section 6.3

Desirable Window Properties

We've seen that to minimize the impact of a window we need the DTFT of the window $W^f(\theta)$ to have:

- Narrow Mainlobe
  - Mainlobe Width *usually* measured "zero-to-zero"

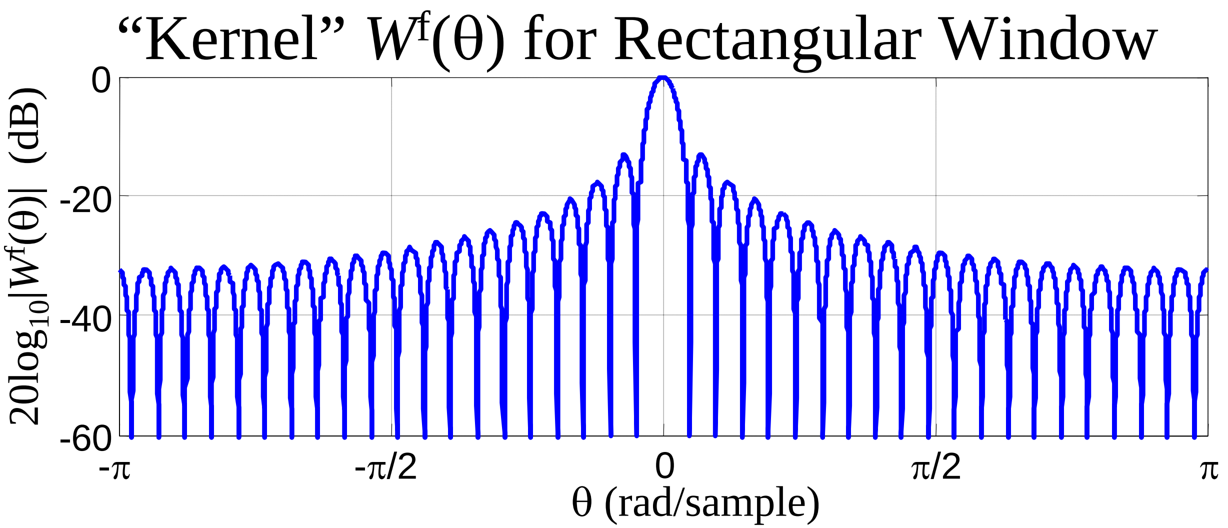
- Small Sidelobe Levels
  - Measured in dB relative to mainlobe peak
  - Care about "Highest Sidelobe" & "Drop-off Rate"

We'll see that there is an inherent trade-off between these two desires:

Lowering the Sidelobes Causes a Widening of the Mainlobe

Rectangular Window

This is what you get if you don't explicitly apply some other type of window – it is due to the fact that you have only *N* signal samples available.

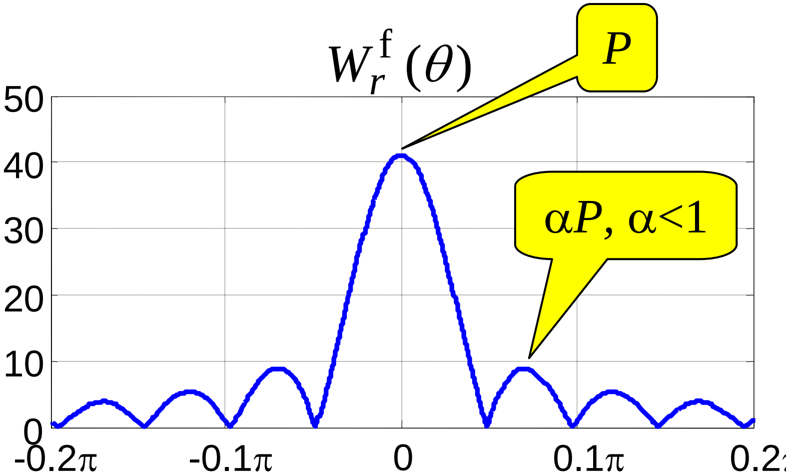


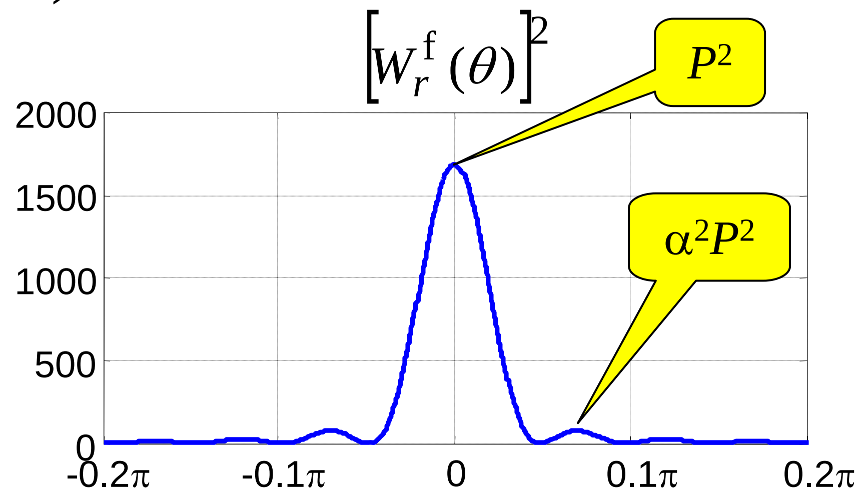
- Mainlobe Width = $4\pi/N$ — Good!!!

- Sidelobe Levels — **Bad!!!** Need to get these lower!!! But HOW????
  - – Largest Sidelobe = –13 dB
  - – Sidelobe Drop-off Rate = –6 dB/octave (except near $\pm\pi$)

Bartlett Window

Inspiration: Square the (non-dB) rect. kernel

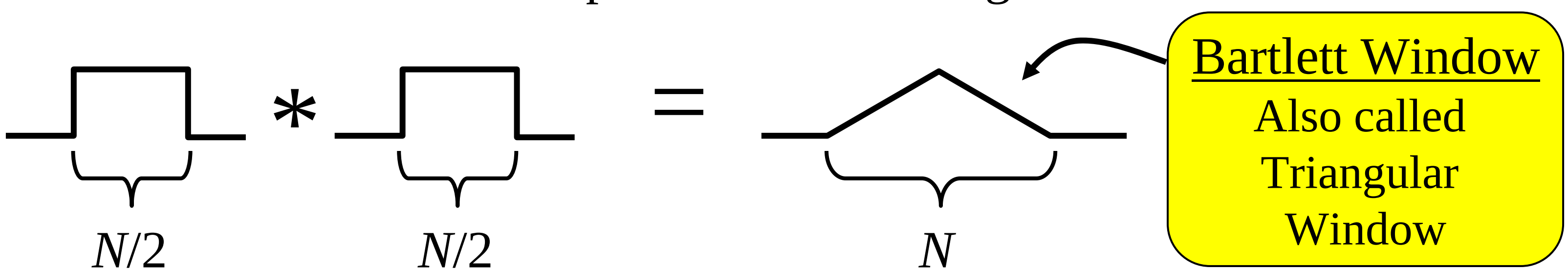




Sidelobe Ratio = $\alpha P/P = \alpha$

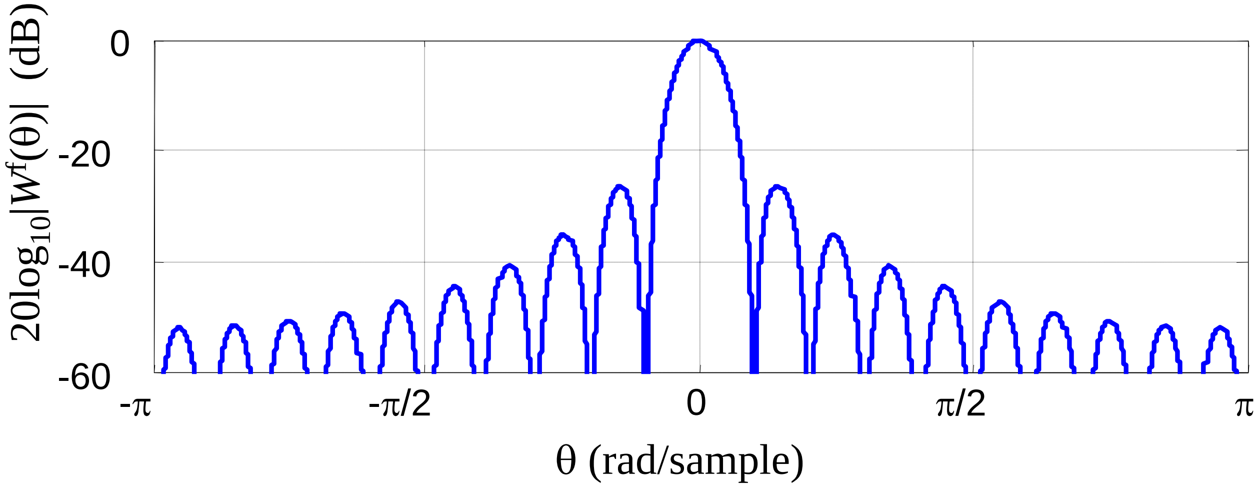
Sidelobe Ratio = $\alpha^2 P^2/P^2 = \alpha^2 < \alpha$

So… in time-domain this corresponds to convolving 2 rect. windows:



Bartlett Window (pt. 2)

"Kernel" $W^f(\theta)$ for Bartlett Window



- <u>Mainlobe Width</u> = $8\pi/(N+1)$

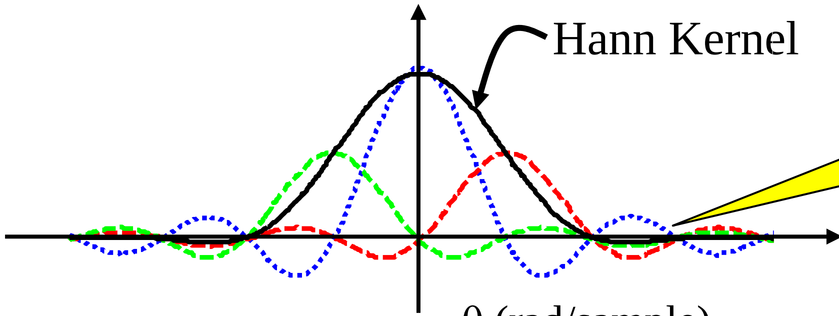
≈ 2 ×Wider Than Rect

- <u>Sidelobe Levels</u>
  - – Largest Sidelobe = –27 dB
  - – Sidelobe Drop-off Rate = –12 dB/octave (except near ± π)

<u>Better than Rect</u>
-27 dB vs. –13 dB
-12 dB/oct vs. -6 dB/oct

Hann Window (also called Hanning)

Inspiration: "Add" three shifted (non-dB) rect. kernels together to try to cancel sidelobes:



$$W_{\text{hn}}^{\text{f}}(\theta) = 0.5W_{\text{r}}^{\text{f}}(\theta) - 0.25W_{\text{r}}^{\text{f}}\left(\theta - \frac{2\pi}{N-1}\right) - 0.25W_{\text{r}}^{\text{f}}\left(\theta + \frac{2\pi}{N-1}\right)$$

Use DTFT Freq. Shift Property

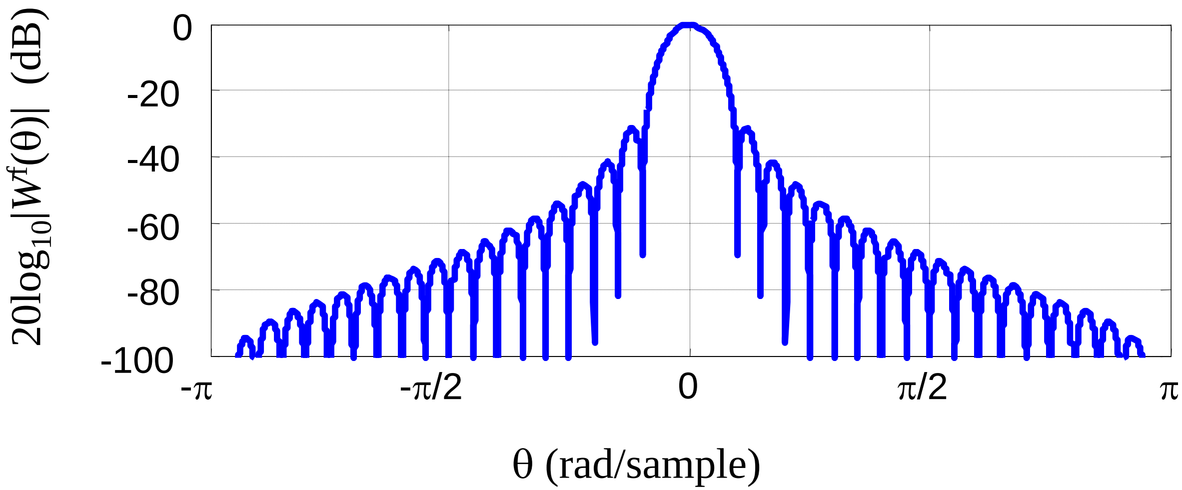
$$w_{\text{hn}}[n] = 0.5 - 0.25e^{j2\pi n/(N-1)} - 0.25e^{-j2\pi n/(N-1)}, \quad 0 \le n \le N-1$$

$$= 0.5\left[1 - \cos\{2\pi n/(N-1)\}\right]$$

Hann Window

Hann Window (pt. 2)

"Kernel" $W^f(\theta)$ for Hann Window



- Mainlobe Width = $8\pi/(N)$

2 ×Wider Than Rect

- Sidelobe Levels
  - – Largest Sidelobe = –32 dB
  - – Sidelobe Drop-off Rate = –18 dB/octave (except near ± π)

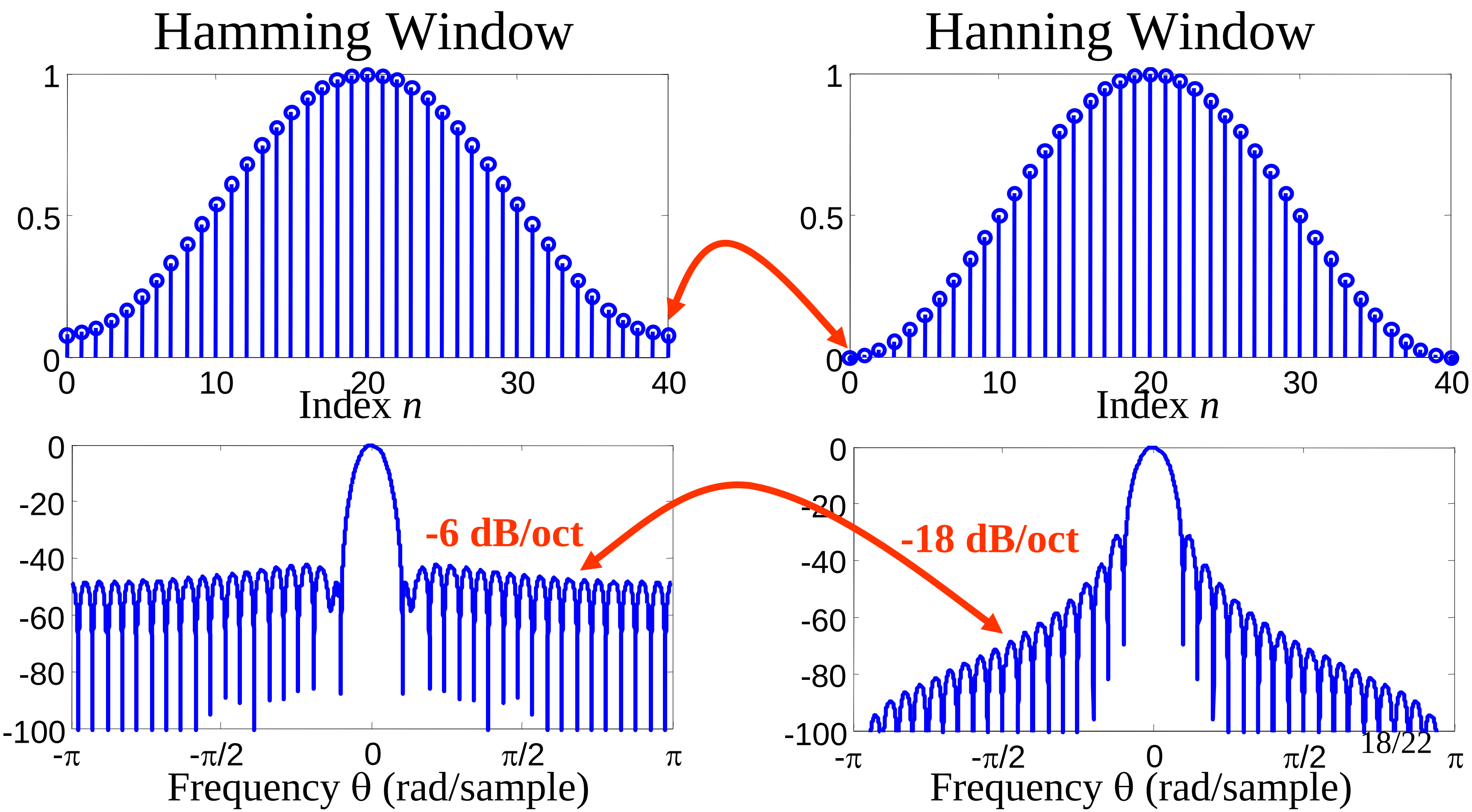
Even Better than Bartlett
-32 dB vs. –27 dB
-18 dB/oct vs. -12 dB/oct

Hamming Window

Inspiration: Tweak Hann coefficients to get lower "highest SL"

$$W_{\mathrm{hm}}^{\mathrm{f}}(\theta) = 0.54 W_{\mathrm{r}}^{\mathrm{f}}(\theta) - 0.23 W_{\mathrm{r}}^{\mathrm{f}}\left(\theta - \frac{2\pi}{N-1}\right) - 0.23 W_{\mathrm{r}}^{\mathrm{f}}\left(\theta + \frac{2\pi}{N-1}\right)$$

$$\boxed{w_{\mathrm{hn}}[n] = 0.54 - 0.46\cos\{2\pi n/(N-1)\}}$$



Hamming Window (pt. 2)

- Mainlobe Width = $8\pi/(N)$

2 ×Wider Than Rect
(same as Hanning)

- Sidelobe Levels
  - Largest Sidelobe = –43 dB
  - Sidelobe Drop-off Rate = –6 dB/octave (except near $\pm\pi$)

Even Better than Hanning
–43 dB vs. –32 dB

As Bad as Rect!!
–6 dB vs. –6 dB

Note: Both Rect & Hamming have –6 dB/oct drop-off

Note also: Both are discontinuous at window edge in time-domain

Drop-Off Rate & Discontinuity Order

Definition: If the window's time-domain function is such that up to its **$(p\text{-}1)^{th}$** derivative (but no higher) is continuous, then we say that the signal has ***p-order continuity***.

Ex. Rectangular Window has 0-order continuity
Triangular Window has 1-order continuity
Hamming Window has 0-order continuity

Result: A window that has *continuity of order p* will (generally) have a kernel that has a sidelobe drop-off rate of $-(p+1)\mathbf{6}$ dB/oct

Rectangular Window has 0-order continuity: - 6dB/oct
Hamming Window has 0-order continuity: - 6dB/oct
Triangular Window has 1-order continuity: - 12dB/oct
Hann Window has 2-order continuity: - 18dB/oct

Other Windows & Their Rationale

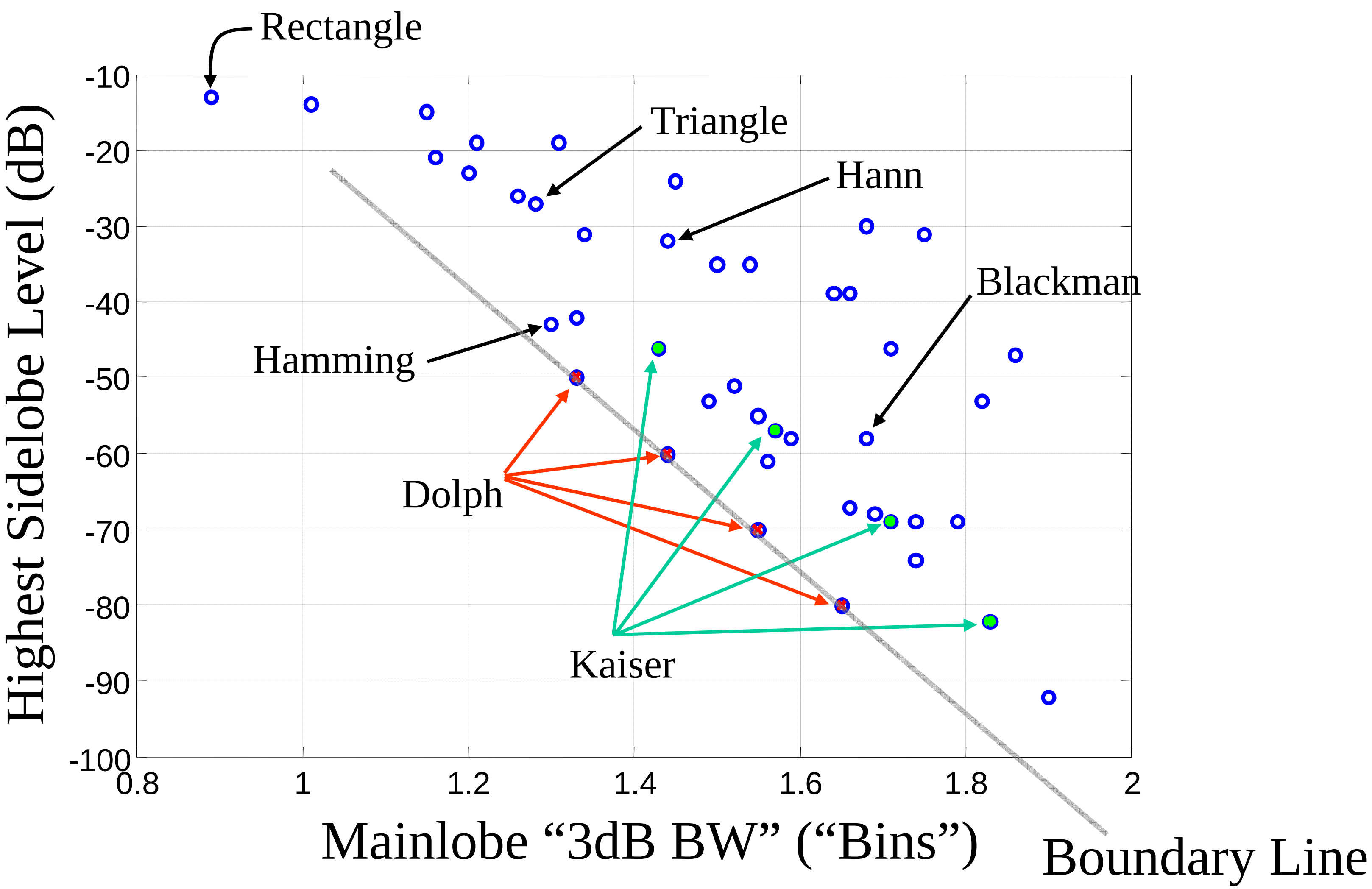
Lots of effort has been focused on designing good windows. Here are a few, with their design rationale and their "specs"

Blackman: "More Tweaking of Hann Coefficients"
ML Width = $12\pi/N$ SL Level = -57 dB Drop-Off = -18 dB/oct

Kaiser: "Minimize width for SL *energy* not exceeding spec'd % of total"
ML Width = variable SL Level = variable Drop-Off = -6 dB/oct

Dolph: "Minimize width for SL *level* not exceeding spec'd level"
ML Width = variable SL Level = variable Drop-Off = 0 dB/oct

Comparison of Windows



Data taken from table in F. J. Harris, “On the use of windows for harmonic analysis with the discrete Fourier transform,” *Proc. IEEE*, vol. 66, pp. 51 – 83, January 1978.