Decimation & Expansion (Frequency Domain View)

Need to look at:

– Z- Transform

– DTFT

M-Fold Decimation – Frequency-Domain

Notation: $\{Zx_{(\downarrow M)}\}(z) = X^{z}_{(\downarrow M)}(z) = \{X^{z}(z)\}_{(\downarrow M)}$

- Similar for DTFT
- Similar for Expansion

Q: What is $X_{(\downarrow M)}(z)$ in terms of $X(z)$???

What do we expect????!!!!

Lower F_s causes Spectral Replicas to Move to Lower Frequencies

Should look exactly like sampling at a lower F_s

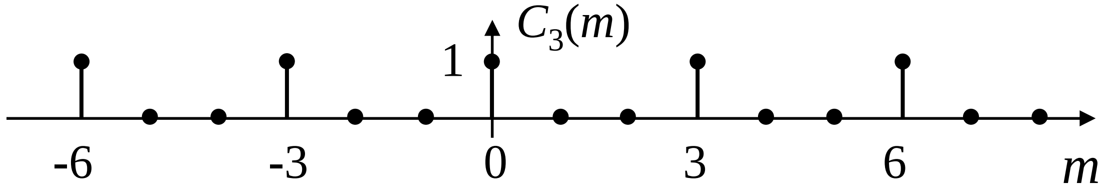
Thus… increased aliasing is possible!!!

To answer this we need to define a useful function ("comb" function):

$$c_M[n] = \sum_{k=-\infty}^{\infty} \delta[n-kM] = \delta[n \bmod M] = \frac{1}{M}\sum_{m=0}^{M-1} W_M^{mn}$$

$$W_M \triangleq e^{j2\pi/M}$$

Call this (★)…
This is like a DT Fourier Series and is easily verified!



M-Fold Decimation – Frequency-Domain (cont.)

Now… use the comb function to write decimation:

$$\begin{aligned} x_{(\downarrow M)}[n] &= x[nM] \\ &= x[nM]c_M[nM] \end{aligned}$$

Doesn't Really Do Anything Here… But Later it Will!!

Now… take Z-Transform, using this form:

$$X^z_{(\downarrow M)}(z) = \underbrace{\sum_{n=-\infty}^{\infty} x[nM]c_M[nM]z^{-n}}_{\cdots + x[0]z^0 + x[M]z^{-1} + x[2M]z^2 + \cdots} = \underbrace{\sum_{n=-\infty}^{\infty} x[n]c_M[n]z^{-n/M}}_{\cdots + x[0]z^0 + \underbrace{0 + \cdots + 0}_{} + x[M]z^{-1} + 0 + \cdots}$$

Action of $C_M[n]$

Now… take Z-Transform, using this form:

$$X^z_{(\downarrow M)}(z) = \sum_{n=-\infty}^{\infty} x[n]\left[\frac{1}{M}\sum_{m=0}^{M-1} W_M^{mn}\right]z^{-n/M}$$

M-Fold Decimation – Frequency-Domain (cont.)

Now… just manipulate:

$$X^{z}_{(\downarrow M)}(z) = \sum_{n=-\infty}^{\infty} x[n]\left[\frac{1}{M}\sum_{m=0}^{M-1} W_M^{mn}\right] z^{-n/M}$$

$$= \frac{1}{M}\sum_{m=0}^{M-1}\left[\sum_{n=-\infty}^{\infty} x[n]\left(W_M^{-m} z^{1/M}\right)^{-n}\right]$$

$$= \frac{1}{M}\sum_{m=0}^{M-1} X^{z}(W_M^{-m} z^{1/M})$$

ZT of Decimated Signal is…

$$X^{z}_{(\downarrow M)}(z) = \frac{1}{M}\sum_{m=0}^{M-1} X^{z}(W_M^{-m} z^{1/M})$$

M-Fold Decimation – Frequency-Domain (cont.)

Now to see a little better what this says… convert ZT to DTFT.

Recall: DTFT is the ZT evaluated on the unit circle:

$$z = e^{j\theta} \quad \Rightarrow \quad z^{1/M} = e^{j\theta/M}$$

Also, by definition: $W_M^{-m} = e^{-j2\pi m/M}$

Then we get…. DTFT of Decimated Signal is…

$$X^{\mathrm{f}}_{(\downarrow M)}(\theta) = \frac{1}{M}\sum_{m=0}^{M-1} X^{\mathrm{f}}\left(\frac{\theta - 2\pi m}{M}\right)$$

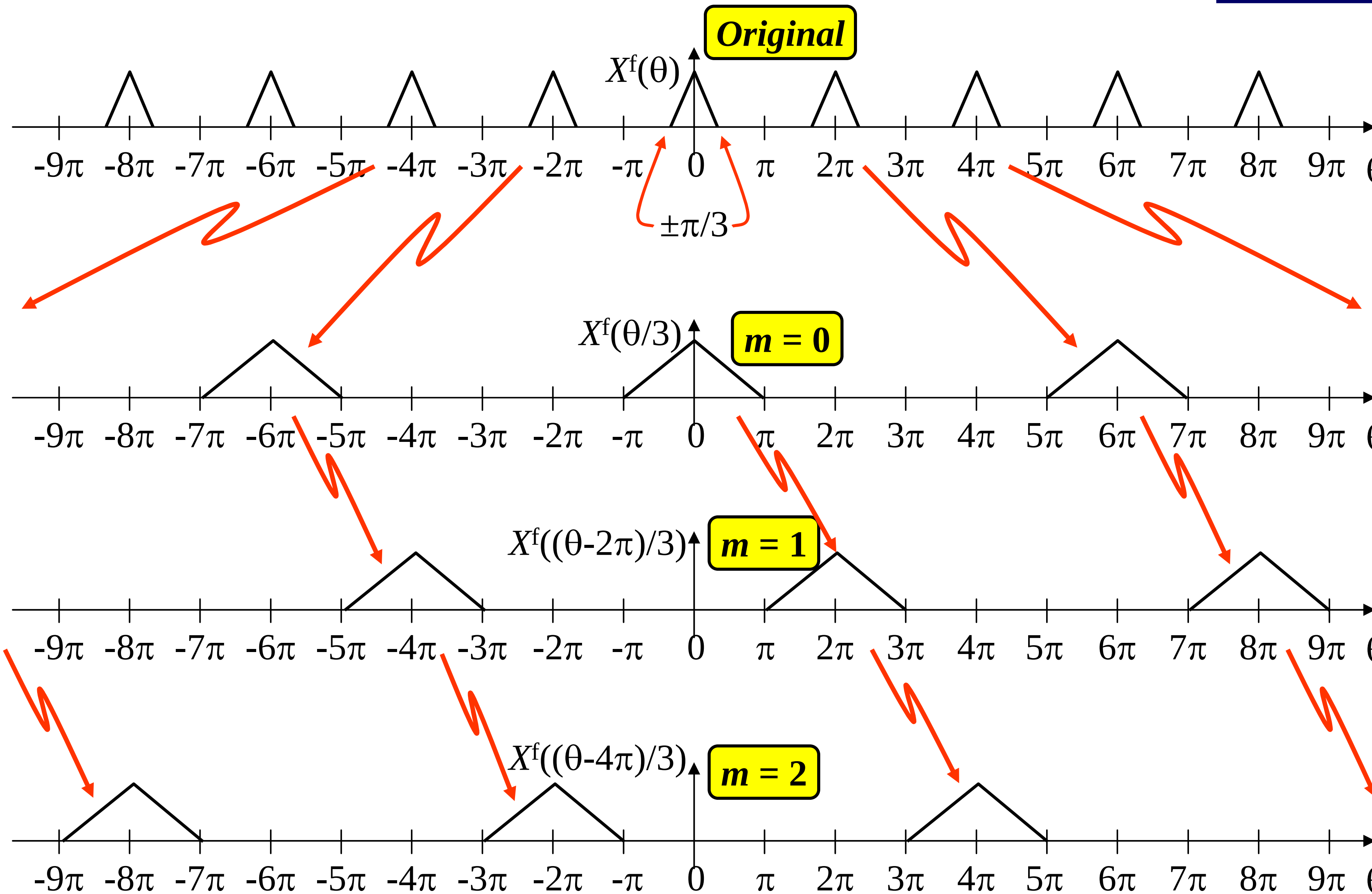
Stretches Spectrum

1. Axis-Scale $X^{\mathrm{f}}(\theta)$ to get $X^{\mathrm{f}}(\theta/M)$ – a stretch
2. Then shift by $2\pi m$ to get new replicas

➔ Decimation Adds Spectral Replicas of Scaled DTFT

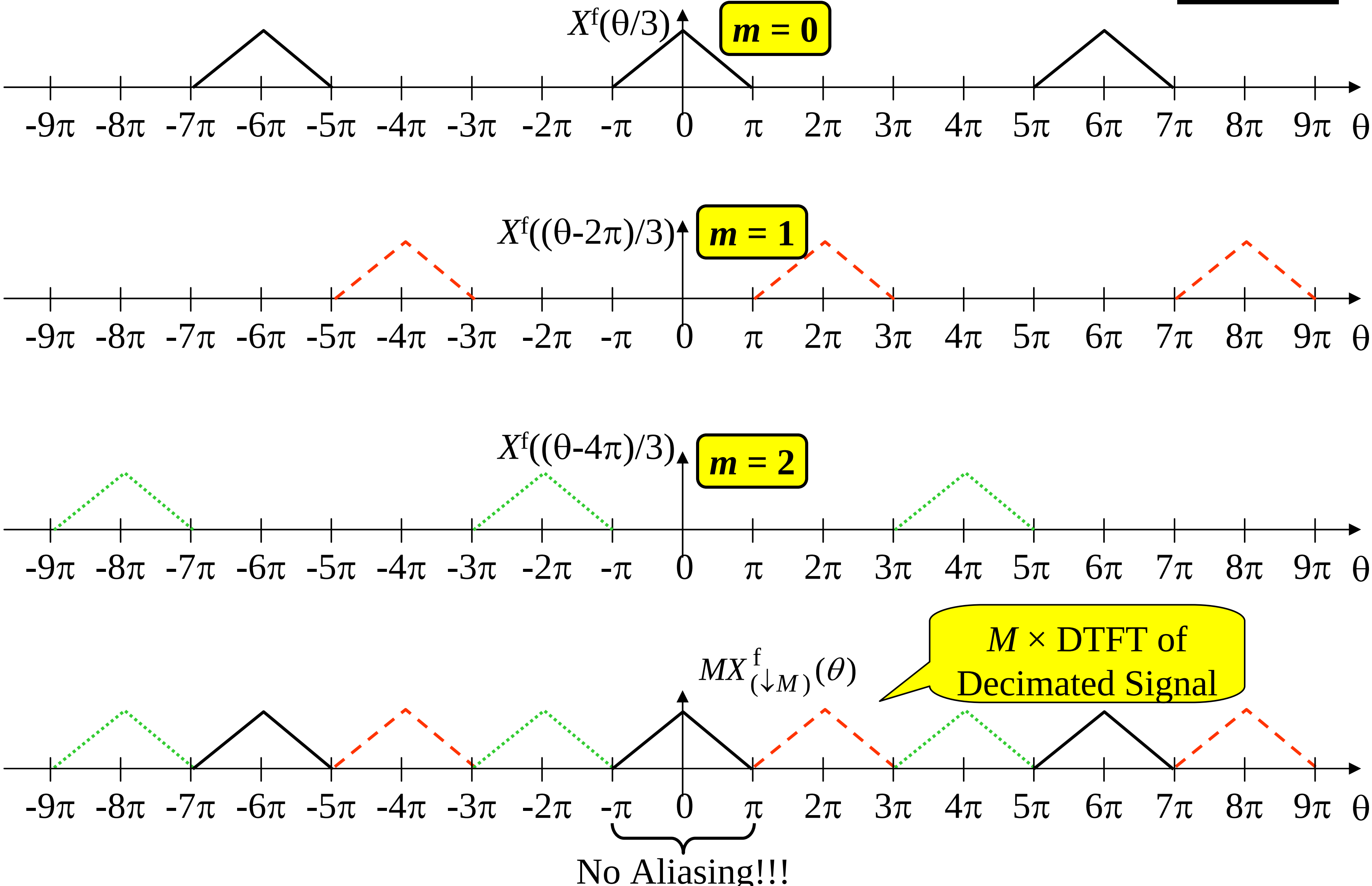
Example: DTFT for M-Fold Decimation

$M = 3$



Example: Continued

M = 3



Example: Insights

1. The *M*-decimated signal will have no aliasing… only if the signal being decimated has: $X^{f}(\theta) = 0 \quad \text{for } |\theta| > \pi / M$

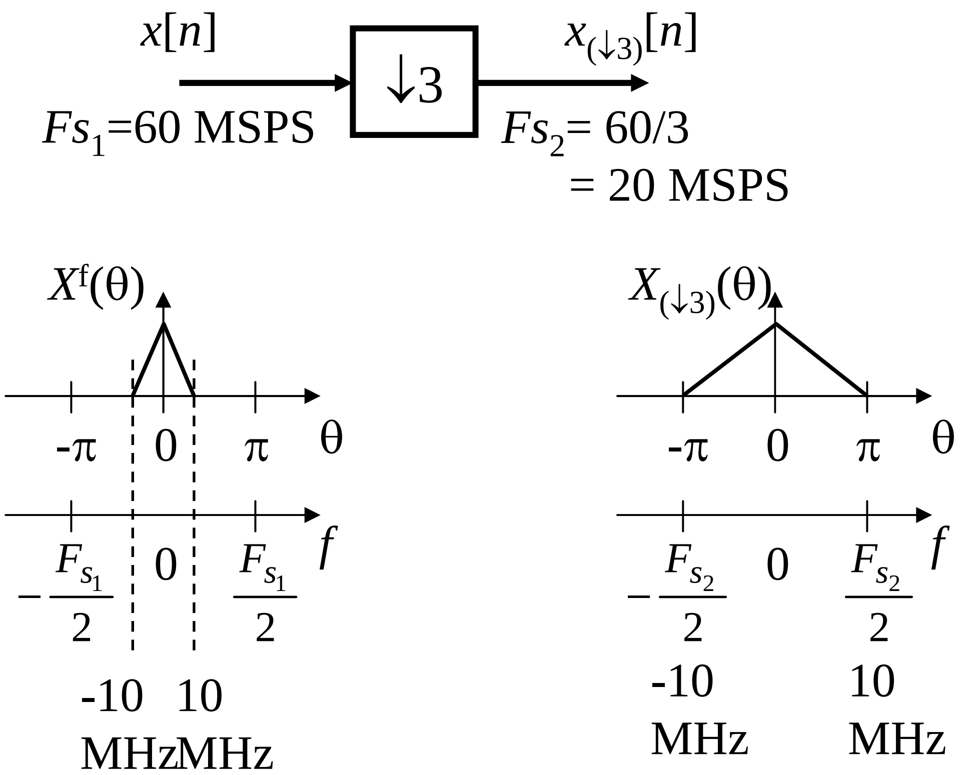
This makes complete sense from an "ordinary" sampling theorem view point!!!

Such a signal is called an "Mth-Band Signal"

2. After *M*-decimating an Mth-band signal, the spectrum of the decimated signal will fill the $[-\pi, \pi]$ band.

Effect on "Physical" Frequency

Although decimation changes the digital frequency of the signal, the corresponding "physical" frequency is not changed… as the following example shows:



<u>Note</u>
Expansion Also Has No Effect on Physical Frequency

Signal Still Occupies **<u>Same</u>** Physical Frequency

L-Fold Expansion – Frequency-Domain

Q: What is $X_{(\uparrow L)}(z)$ in terms of $X(z)$???

What do we expect????!!!!

Certainly **NOT** the same as *really* sampling at a higher rate because of the inserted zeros!!!

Frequency Domain analysis answers this!!!

$$X^{z}_{(\uparrow L)}(z) = \sum_{n=-\infty}^{\infty} x_{(\uparrow L)}[n] z^{-n}$$

$$= + \cdots + x[0]z^{0} + \underbrace{0 + \cdots + 0}_{L-1 \text{ zeros}} + x[1]z^{-L} + \underbrace{0 + \cdots + 0}_{L-1 \text{ zeros}} + x[2]z^{-2L}$$

$$= \sum_{n=-\infty}^{\infty} x[n] z^{-Ln} = X^{z}(z^{L})$$

ZT of Expanded Signal is…

$$X^{z}_{(\uparrow L)}(z) = X^{z}(z^{L})$$

L-Fold Expansion – Frequency-Domain (cont.)

Now to see a little better what this says… convert ZT to DTFT.

Recall: DTFT is the ZT evaluated on the unit circle:

$$z = e^{j\theta} \quad \Rightarrow \quad z^L = e^{jL\theta}$$

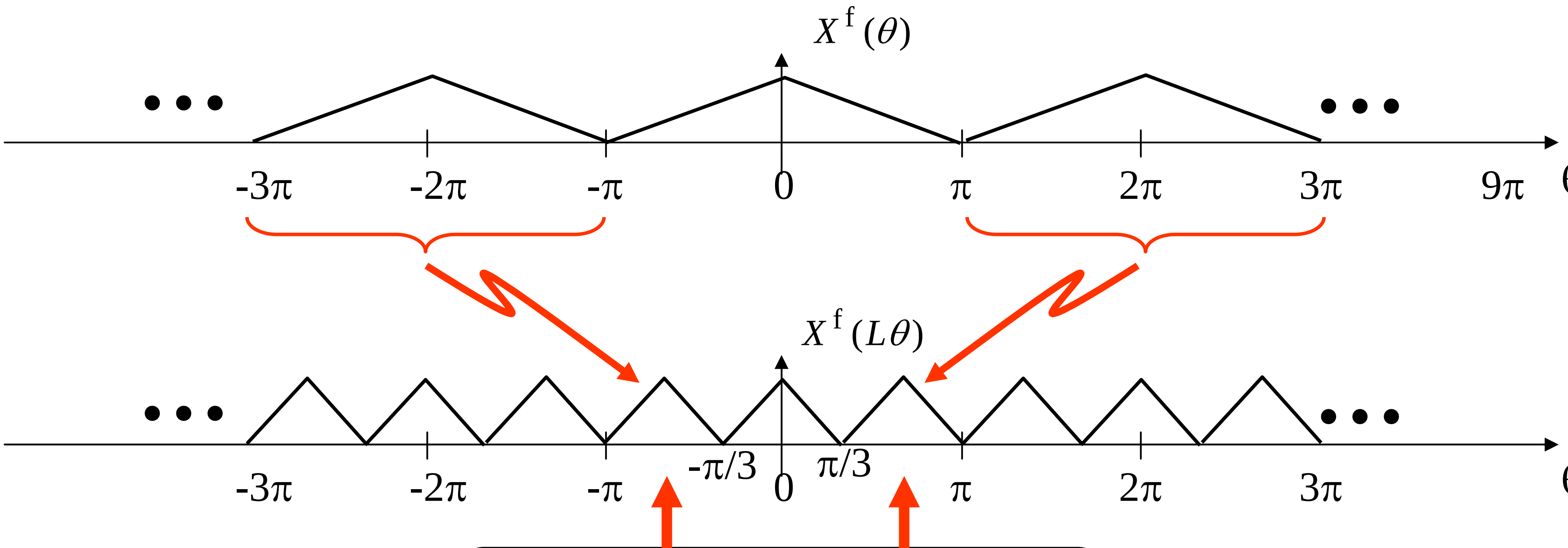
DTFT of Decimated Signal is…

$$X^{\mathrm{f}}_{(\uparrow L)}(\theta) = X^{\mathrm{f}}(L\theta)$$

Scrunches Spectrum

Example: DTFT for *L*-Fold Expansion

$L = 3$



Expansion Causes Images
to Appear in the [-π,π] Range

Here's what we'd have if we <u>REALLY</u> sampled
3 times as fast… <u>**No Images**</u>!!!