Better Proof that FIR QMF gives PR only when it is a two-tap filter

(A)

$$F(z) = \left[G_0(z) + G_0(-z)\right]\left[G_0(z) - G_0(z)\right]$$

Consider case of filter order N is odd (can similarly handle even case)

Then $\quad G_0(z) = c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_N z^{-N}$

and $\quad G_0(-z) = c_0 - c_1 z^{-1} + c_2 z^{-2} - c_3 z^{-3} + \cdots - c_N z^{-N}$

So:

$$G_{even}(z) \triangleq \left[G_0(z) + G_0(-z)\right] = 2\left[c_0 + c_2 z^{-2} + c_4 z^{-4} + \cdots + c_{N-1} z^{-(N-1)}\right]$$

$$G_{odd}(z) \triangleq \left[G_0(z) - G_0(-z)\right] = 2\left[c_1 z^{-1} + c_3 z^{-3} + c_5 z^{-5} + \cdots + c_N z^{-N}\right]$$

↑ indicates which powers are present

So $\quad F(z) = G_{even}(z)\, G_{odd}(z)$

and for PR we want: $\quad G_{even}(z)\, G_{odd}(z) = c\, z^{-\ell}$

"want"

or

$$\left[c_0 + c_2 z^{-2} + c_4 z^{-4} + \cdots + c_{N-1} z^{-(N-1)}\right]\underbrace{\left[c_1 z^{-1} + c_3 z^{-3} + \cdots + c_N z^{-N}\right]}_{\text{factor out } z^{-1} \text{ and swing to other side}} \overset{?}{=} \tilde{c}\, z^{-\ell}$$

$$\Rightarrow \underbrace{\left[c_0 + c_2 z^{-2} + c_4 z^{-4} + \cdots + c_{N-1} z^{-(N-1)}\right]}_{\triangleq D_1(z^2)}\underbrace{\left[c_1 + c_3 z^{-2} + \cdots + c_N z^{-(N-1)}\right]}_{\triangleq D_2(z^2)} \overset{?}{=} \tilde{c}\, z^{-\tilde{\ell}}$$

(B)

Now replace z by $z^{1/2}$ to get

$$D_1(z)\, D_2(z) \stackrel{?}{=} \tilde{c}\, z^{-\tilde{\ell}/2}$$

We now see that $\tilde{\ell}$ <u>must</u> be even since we have only integer powers on left-hand side

$\Rightarrow$ Let $\tilde{\ell} = 2p$

So $$\underset{\substack{\uparrow \\ \text{FIR}}}{D_1(z)}\ \underset{\substack{\uparrow \\ \text{FIR}}}{D_2(z)} \stackrel{?}{=} \tilde{c}\, z^{-p}$$

Now $$D_1(z) \stackrel{?}{=} \frac{\tilde{c}\, z^{-p}}{D_2(z)} = \frac{\tilde{c}\, z^{-p}}{c_1 + c_3 z^{-1} + c_5 z^{-2} + \cdots + c_N z^{-(N-1)}}$$

FIR $\neq$ IIR (in general)

This is not IIR only when <u>all but one</u> of $\{c_1, c_3, \ldots, c_N\}$ are zero.

$\Rightarrow D_2(z)$ has <u>one</u> tap

$\Rightarrow D_1(z)$ " " "

which means that ~~only~~ there are only <u>two</u> <u>non-zero</u> <u>elements</u> in $\{c_0, c_1, c_2, \ldots, c_N\}$

Recall: $G_0(z) = c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_N z^{-N}$

so $G_0(z)$ has <u>only two non-zero taps</u>

Q.E.D.