Chapter 2

Minimum Variance Unbiased Estimators

Ch. 2: Minimum Variance Unbiased Est.

MVU

Basic Idea of MVU: Out of all unbiased estimates, find the one with the lowest variance

(This avoids the realizability problem of MSE)

2.3 Unbiased Estimators

An estimator is unbiased if

$$E\left\{\hat{\theta}\right\} = \theta \quad \text{for all} \quad \theta$$

Example: Estimate DC in White Uniform Noise

$$x[n] = A + w[n] \qquad n = 0,1, \ldots, N-1$$

Unbiased Estimator:

$$\hat{A} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

$$\textit{same as before}: E\{\hat{A}\} = A \quad \textit{regardless of } A \textit{ value}$$

Biased Estimator:

$$\overset{\vee}{A} \frac{1}{N} \sum_{n>0}^{N-1} \left|x(n)\right|$$

$$Note: if \;\; A \geq 1, then \left|x[n]\right| = x[n]$$

$$\Rightarrow \quad \overset{\vee}{A} = \hat{A} \qquad \Rightarrow E\left\{\overset{\vee}{A}\right\} = A$$

$$if \;\; A < 1 \, , then \;\; E\left\{\overset{\vee}{A}\right\} \neq A$$

$$\Rightarrow Bias \begin{cases} = 0 \;\; if \;\; A \geq 1 \\ \\ \neq 0 \;\; if \;\; A < 1 \end{cases} \Rightarrow \quad \text{Biased Est.}$$

2.4 Minimum Variance Criterion

(Recall problem with MMSE criteria)

Constrain bias to be zero 0 find the estimator that minimizes variance

Note:

$$mse\ (\hat{\theta}) = \mathrm{var}(\hat{\theta}) + b^2(\hat{\theta})$$

$b^2(\hat{\theta})$ = 0 for MVU

So, MVU could also be called

"Minimum MSE Unbiased Est."

MVUE = Minimum Variance Unbiased Estimator

2.5 Existence of MVU Estimator

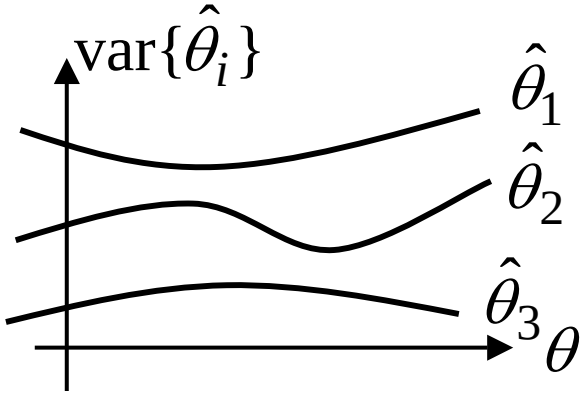
Sometimes there is no MVUE… can happen 2 ways:

1. There may be no unbiased estimators
2. None of the above unbiased estimators has a uniformly minimum variance

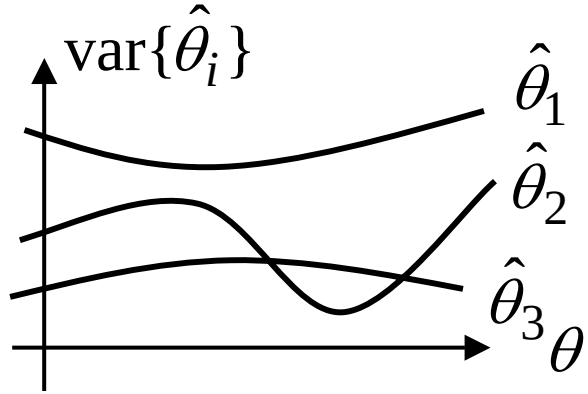
Ex. of #2 $\hat{\theta}_i = g_i(\mathbf{x}), \quad i = 1, 2, 3$

Assume there are only 3 unbiased estimators for a problem. Two possible cases:

$\exists$ an MVU



$\nexists$ an MVU



2.6 Finding the MVU Estimator

Even if MVU exists: may not be able to find it!!

No Known "turn the crank" Method

Three Approaches to Finding the MVUE

1. Determine Cramer-Rao Lower Bound (CRLB)
   … and see if some estimator satisfies it (Ch 3 & 4)
   (Note: MVU can exist but not achieve the CRLB)

2. Apply Rao-Blackwell-Lechman-Scheffe Theorem
   Rare in Practice… We'll skip Ch. 5

3. Restrict to Linear Unbiased & find MVLU (Ch. 6)
   Only gives true MVU if problem is linear

2.7 Vector Parameter

When we wish to estimate multiple parameters we group them into a vector:

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_p \end{bmatrix}^T$$

Then an estimator is notated as: $\hat{\boldsymbol{\theta}} = \begin{bmatrix} \hat{\theta}_1 & \hat{\theta}_2 & \cdots & \hat{\theta}_p \end{bmatrix}^T$

Unbiased requirement becomes:

$$E\left\{\hat{\boldsymbol{\theta}}\right\} = \boldsymbol{\theta}$$

Minimum Variance requirement becomes:

For each *i*…

$$\text{var}\left\{\hat{\boldsymbol{\theta}}\right\} = \boldsymbol{\theta} \quad \text{min over all unbiased estimates}$$