3.8 Vector Transformations

Just like for the scalar case…. $\alpha = g(\theta)$

If you know CRLB_θ you can find CRLB_α

$$\mathbf{CRLB}_{\boldsymbol{\alpha}} = \underbrace{\left[\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right] \underbrace{\mathbf{I}^{-1}(\boldsymbol{\theta})}_{CRLB\ on\ \boldsymbol{\theta}} \left[\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right]^T}_{CRLB\ on\ \boldsymbol{\alpha}}$$

Jacobian Matrix (see p. 46)

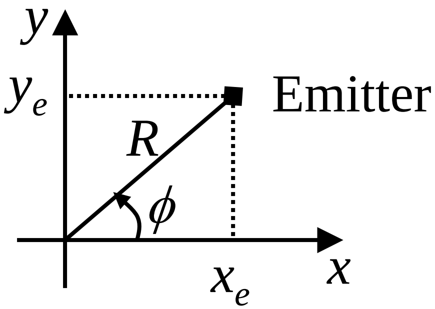
Example: Usually can estimate Range (R) and Bearing (φ) directly

But might really want emitter (x, y)

Example of Vector Transform

Can estimate Range (R) and Bearing (ϕ) directly

But might really want emitter location (x_e, y_e)



Direct Parameters

$$\boldsymbol{\theta} = \begin{bmatrix} R \\ \phi \end{bmatrix}$$

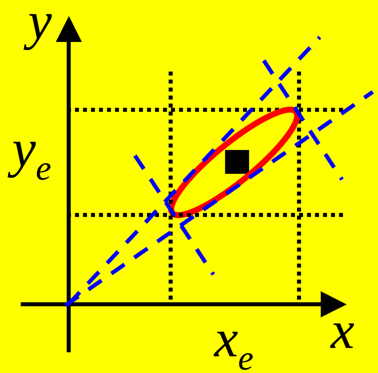
$$\boldsymbol{\alpha} = \begin{bmatrix} x_e \\ y_e \end{bmatrix} = g(\boldsymbol{\theta}) = \begin{bmatrix} R\cos\phi \\ R\sin\phi \end{bmatrix}$$

Mapped Parameters

Jacobian Matrix

$$\frac{\partial g(\theta)}{\partial \theta} = \begin{bmatrix} \dfrac{\partial R\cos\phi}{\partial R} & \dfrac{\partial R\cos\phi}{\partial \phi} \\ \dfrac{\partial R\sin\phi}{\partial R} & \dfrac{\partial R\cos\phi}{\partial \phi} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -R\sin\phi \\ \sin\phi & R\cos\phi \end{bmatrix}$$



$$\mathbf{CRLB}_{\boldsymbol{\alpha}} = \left[\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right] \mathbf{CRLB}_{\boldsymbol{\theta}} \left[\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right]^T$$

3.9 CRLB for General Gaussian Case

In Sect. 3.5 we saw the CRLB for "signal + AWGN"

Deterministic Signal w/
Scalar Deterministic Parameter

For that case we saw:

The PDF's parameter-dependence showed up only in the mean of the PDF

Now generalize to the case where:

$$\mathbf{x} \sim N\left(\boldsymbol{\mu}(\boldsymbol{\theta}), \mathbf{C}(\boldsymbol{\theta})\right)$$

- Data is still Gaussian, but
- Parameter-Dependence not restricted to Mean
- Noise not restricted to White… Cov not necessarily diagonal

One way to get this case: "signal + AGN"

Random Gaussian Signal w/
Vector Deterministic Parameter

Non-White Noise

For this case the FIM is given by: (See Appendix 3c)

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = \underbrace{\left[\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_i}\right]^T \mathbf{C}^{-1}(\boldsymbol{\theta}) \left[\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_j}\right]} + \underbrace{\frac{1}{2} tr\left[\mathbf{C}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{C}(\boldsymbol{\theta})}{\partial \theta_i} \mathbf{C}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{C}(\boldsymbol{\theta})}{\partial \theta_j}\right]}$$

Variability of Mean w.r.t. parameters

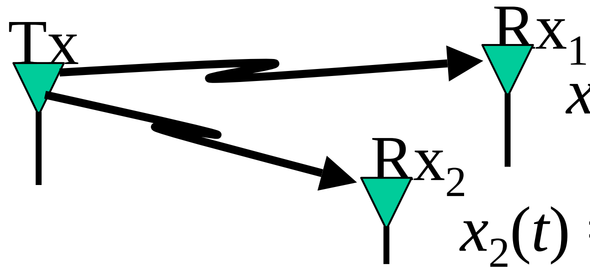
This shows the impact of signal *model* assumptions

- deterministic signal + AGN
- random Gaussian signal + AGN

Est. Cov. uses average over only noise

Est. Cov. uses average over signal & noise

Gen. Gauss. Ex.: Time-Difference-of-Arrival



$$x_1(t) = s(t - \Delta\tau) + w_1(t)$$

$$x_2(t) = s(t) + w_2(t)$$

Given: $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T \end{bmatrix}^T$

Goal: Estimate $\Delta\tau$

How to model the signal?

- Case #1: $s(t)$ is zero-mean, WSS, Gauss. Process — Passive Sonar
- Case #2: $s(t)$ is a deterministic signal — Radar/Comm Location

Case #1

$\boldsymbol{\mu}(\Delta\tau) = \mathbf{0}$ ➡ **No Term #1**

$$\mathbf{C}(\Delta\tau) = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12}(\Delta\tau) \\ \mathbf{C}_{21}(\Delta\tau) & \mathbf{C}_{22} \end{bmatrix}$$

$$\mathbf{C}_{ii} = \mathbf{C}_{\mathbf{s}_i\mathbf{s}_i} + \mathbf{C}_{\mathbf{w}_i\mathbf{w}_i}$$

$$\mathbf{C}_{ij}(\Delta\tau) = \mathbf{C}_{\mathbf{s}_i\mathbf{s}_j}(\Delta\tau)$$

Case #1

$\mathbf{C}(\Delta\tau) = \mathbf{C}$ ➡ **No Term #2**

$$\boldsymbol{\mu}(\Delta\tau) = \begin{bmatrix} s_1[0;\Delta\tau] \\ s_1[1;\Delta\tau] \\ \vdots \\ s_1[N-1;\Delta\tau] \\ s_2[0;\Delta\tau] \\ \vdots \\ s_2[N-1;\Delta\tau] \end{bmatrix}$$

Comments on General Gaussian CRLB

It is interesting to note that for any given problem you may find each case used in the literature!!!

For example for the TDOA/FDOA estimation problem:

- Case #1 used by M. Wax in *IEEE Trans. Info Theory*, Sept. 1982
- Case #2 used by S. Stein in *IEEE Trans. Signal Proc.*, Aug. 1993

See also differences in the book's examples

We'll skip Section 3.10 and leave it as a reading assignment