3.11 CRLB Examples

We'll now apply the CRLB theory to several examples of practical signal processing problems.

We'll revisit these examples in Ch. 7… we'll derive ML estimators that will get close to achieving the CRLB

1. Range Estimation
   - sonar, radar, robotics, emitter location
2. Sinusoidal Parameter Estimation (Amp., Frequency, Phase)
   - sonar, radar, communication receivers (recall DSB Example), etc.
3. Bearing Estimation
   - sonar, radar, emitter location
4. Autoregressive Parameter Estimation
   - speech processing, econometrics

Ex. 1 Range Estimation Problem

Transmit Pulse: $s(t)$ nonzero over $t \in [0, T_s]$

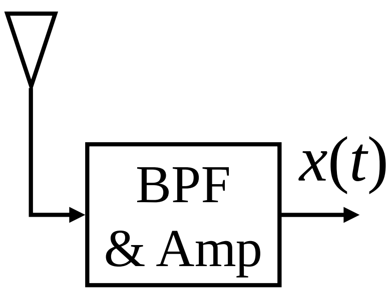
Receive Reflection: $s(t - \tau_o)$

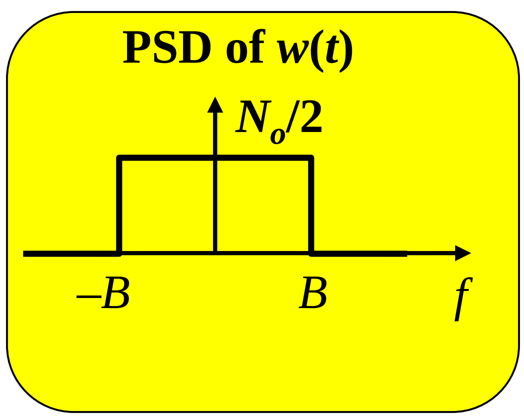
Measure Time Delay: τ_o

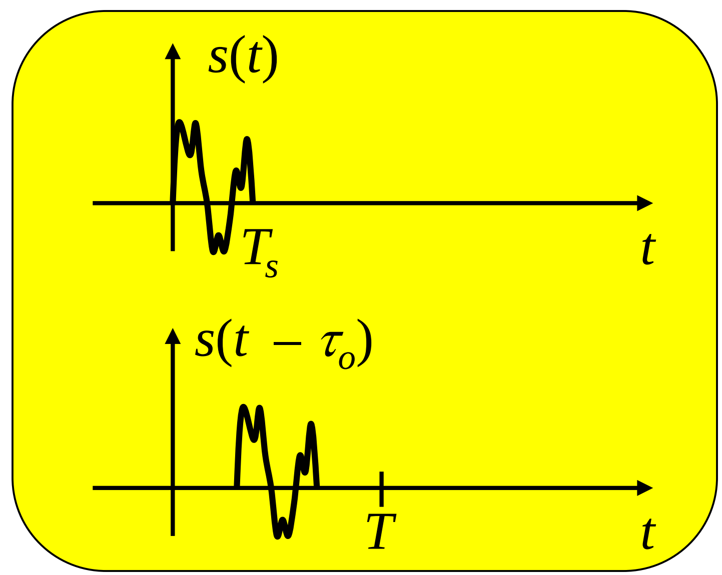
C-T Signal Model

$$x(t) = \underbrace{s(t - \tau_o)}_{s(t;\tau_o)} + w(t) \qquad 0 \le t \le T = T_s + \tau_{o,\max}$$

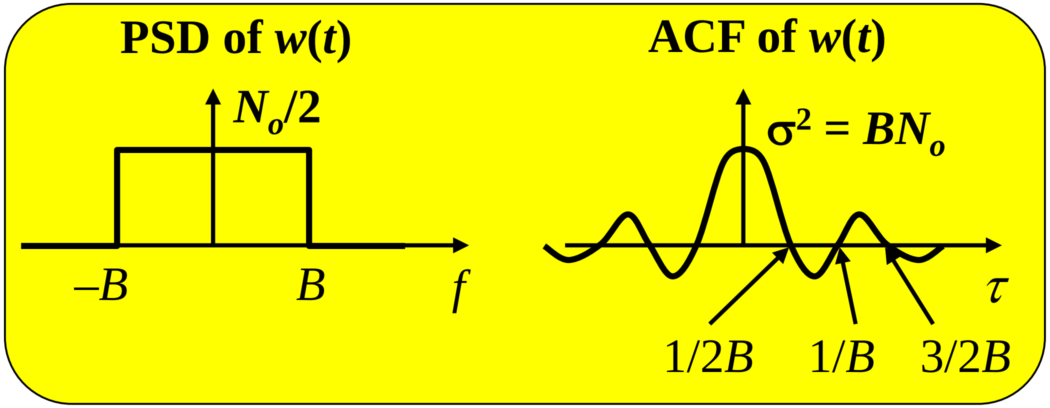
Bandlimited White Gaussian







Range Estimation D-T Signal Model



Sample Every $\Delta = 1/2B$ sec

$$w[n] = w(n\Delta)$$

DT White Gaussian Noise
Var $\sigma^2 = BN_o$

$$x[n] = \underbrace{s(n\Delta - \tau_o)}_{} + w[n] \qquad n = 0, 1, \ldots, N-1$$

$s[n;\tau_o]$… has M non-zero samples starting at n_o

$n_o = \tau_o / \Delta$

$$x[n] = \begin{cases} w[n] & 0 \le n \le n_o - 1 \\ s(n\Delta - \tau_o) + w[n] & n_o \le n \le n_o + M - 1 \\ w[n] & n_o + M \le n \le N - 1 \end{cases}$$

Range Estimation CRLB

Now apply standard CRLB result for signal + WGN:

Plug in… and keep non-zero terms

$$\operatorname{var}(\hat{\tau}_o) \geq \frac{\sigma^2}{\sum_{n=0}^{N-1}\left(\frac{\partial s[n;\tau_o]}{\partial \tau_o}\right)^2} = \frac{\sigma^2}{\sum_{n=n_o}^{n_o+M-1}\left(\frac{\partial s(n\Delta-\tau_o)}{\partial \tau_o}\right)^2}$$

$$= \frac{\sigma^2}{\sum_{n=n_o}^{n_o+M-1}\left(\left.\frac{\partial s(t)}{\partial t}\right|_{t=n\Delta-\tau_o}\right)^2} = \frac{\sigma^2}{\sum_{n=0}^{M-1}\left(\left.\frac{\partial s(t)}{\partial t}\right|_{t=n\Delta}\right)^2}$$

Exploit Calculus!!!

Use approximation: $\tau_o = \Delta\, n_o$
Then do change of variables!!

Range Estimation CRLB (cont.)

Assume sample spacing is small… approx. sum by integral…

$$\mathrm{var}(\hat{\tau}_o) \geq \frac{\sigma^2}{\frac{1}{\Delta}\int_0^{T_s}\left(\frac{\partial s(t)}{\partial t}\right)^2 dt} = \frac{N_o/2}{\int_0^{T_s}\left(\frac{\partial s(t)}{\partial t}\right)^2 dt} = \frac{1}{\frac{E_s}{N_o/2}\,\frac{\int_0^{T_s}\left(\frac{\partial s(t)}{\partial t}\right)^2 dt}{E_s}}$$

$$E_s = \int_0^{T_s} s^2(t)\,dt$$

FT Theorem & Parseval

$$\mathrm{var}(\hat{\tau}_o) \geq \frac{1}{\frac{E_s}{N_o/2}\,\frac{\int_0^{T_s}(2\pi f)^2|S(f)|^2 df}{E_s}}$$

$$= \frac{1}{\frac{E_s}{N_o/2}\,\frac{\int_{-\infty}^{\infty}(2\pi f)^2|S(f)|^2 df}{\int_{-\infty}^{\infty}|S(f)|^2 dt}}$$

Parseval

A type of "SNR"

Define a BW measure:

$$B_{rms} = \sqrt{\frac{\int_{-\infty}^{\infty}(2\pi f)^2|S(f)|^2 df}{\int_{-\infty}^{\infty}|S(f)|^2 dt}}$$

B_{rms} is "RMS BW" (Hz)

Range Estimation CRLB (cont.)

Using these ideas we arrive at the CRLB on the delay:

$$\operatorname{var}(\hat{\tau}_o) \geq \frac{1}{SNR_E \times B_{rms}^2} \quad (\sec^2)$$

This "SNR" is not our usual ratio of powers… so let's convert to our usual form:

$$E_s = \int_0^{T_s} s^2(t)\,dt \quad \Rightarrow \quad P_s = \frac{1}{T_s}\int_0^{T_s} s^2(t)\,dt = \frac{E_s}{T_s}$$

$$P_n = \frac{N_o}{2} \times (2B)$$

Thus…

$$SNR = \frac{P_s}{P_n} = \frac{E_s / T_s}{\frac{N_o}{2} \times (2B)} \quad \Rightarrow \quad SNR_E = 2BT_s SNR$$

$$\Rightarrow \quad \operatorname{var}(\hat{\tau}_o) \geq \frac{1}{2BT_s SNR \times B_{rms}^2} \quad (\sec^2)$$

Range Estimation CRLB (cont.)

To get the CRLB on the **range**… use "transf. of parms" result:

$$CRLB_{\hat{R}} = \left(\frac{\partial R}{\partial \tau_o}\right)^2 CRLB_{\hat{\tau}_o} \quad \text{with} \quad R = c\tau_o / 2$$

$$\operatorname{var}\left(\hat{R}\right) \geq \frac{c^2/4}{2BT_s SNR \times B_{rms}^2} \quad \left(m^2\right)$$

CRLB is inversely proportional to:

- *SNR* Measure
- RMS BW Measure

So the CRLB tells us…

- Choose signal with large B_{rms}
- Ensure that SNR is large
- Better on Nearby/large targets
- Which is better?
  - Double transmitted energy/power?
  - Double RMS bandwidth?

Ex. 2 Sinusoid Estimation CRLB Problem

Given DT signal samples of a sinusoid in noise….

Estimate its amplitude, frequency, and phase

$$x[n] = A\cos(\Omega_o n + \phi) + w[n] \qquad n = 0, 1, \ldots, N-1$$

Ω_o is DT frequency in rad/sample: $0 < \Omega_o < \pi$

DT White Gaussian Noise Zero Mean & Variance of σ^2

Multiple parameters… so parameter vector: $\boldsymbol{\theta} = [A \quad \Omega_o \quad \phi]^T$

Recall… SNR of sinusoid in noise is:

$$SNR = \frac{P_s}{P_n} = \frac{A^2/2}{\sigma^2} = \frac{A^2}{2\sigma^2}$$

Sinusoid Estimation CRLB Approach

Approach:

- Find Fisher Info Matrix
- Invert to get CRLB matrix
- Look at diagonal elements to get bounds on parm variances

Recall: Result for FIM for general Gaussian case specialized to signal in AWGN case:

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = \frac{1}{\sigma^2}\left(\frac{\partial \mathbf{s}_{\boldsymbol{\theta}}}{\partial \theta_i}\right)\left(\frac{\partial \mathbf{s}_{\boldsymbol{\theta}}}{\partial \theta_j}\right)^T$$

$$= \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\frac{\partial s[n;\boldsymbol{\theta}]}{\partial \theta_i}\frac{\partial s[n;\boldsymbol{\theta}]}{\partial \theta_j}$$

Sinusoid Estimation Fisher Info Elements

Taking the partial derivatives and using approximations given in book (valid when Ω_o is not near 0 or π) :

$$\boldsymbol{\theta} = [A \quad \Omega_o \quad \phi]^T$$

$$[\mathbf{I}(\boldsymbol{\theta})]_{11} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\cos^2(\Omega_o n + \phi) = \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(1 + \cos(2\Omega_o n + 2\phi)\right) \approx \frac{N}{2\sigma^2}$$

$$[\mathbf{I}(\boldsymbol{\theta})]_{12} = [\mathbf{I}(\boldsymbol{\theta})]_{21} = \frac{-1}{\sigma^2}\sum_{n=0}^{N-1}An\cos(\Omega_o n + \phi)\sin(\Omega_o n + \phi) = \frac{-A}{2\sigma^2}\sum_{n=0}^{N-1}n\sin(2\Omega_o n + 2\phi) \approx 0$$

$$[\mathbf{I}(\boldsymbol{\theta})]_{13} = [\mathbf{I}(\boldsymbol{\theta})]_{31} = \frac{-1}{\sigma^2}\sum_{n=0}^{N-1}A\cos(\Omega_o n + \phi)\sin(\Omega_o n + \phi) = \frac{-A}{2\sigma^2}\sum_{n=0}^{N-1}\sin(2\Omega_o n + 2\phi) \approx 0$$

$$[\mathbf{I}(\boldsymbol{\theta})]_{22} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}A^2(n)^2\sin^2(\Omega_o n + \phi) = \frac{A^2}{2\sigma^2}\sum_{n=0}^{N-1}n^2\left(1 - \cos(2\Omega_o n + 2\phi)\right) \approx \frac{A^2}{2\sigma^2}\sum_{n=0}^{N-1}n^2$$

$$[\mathbf{I}(\boldsymbol{\theta})]_{23} = [\mathbf{I}(\boldsymbol{\theta})]_{32} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}A^2 n\sin^2(\Omega_o n + \phi) \approx \frac{A^2}{2\sigma^2}\sum_{n=0}^{N-1}n$$

$$[\mathbf{I}(\boldsymbol{\theta})]_{33} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}A^2\sin^2(\Omega_o n + \phi) \approx \frac{NA^2}{2\sigma^2}$$

Sinusoid Estimation Fisher Info Matrix

$$\boldsymbol{\theta} = [A \quad \Omega_o \quad \phi]^T$$

Fisher Info Matrix then is:

$$\mathbf{I}(\boldsymbol{\theta}) \approx \begin{bmatrix} \dfrac{N}{2\sigma^2} & 0 & 0 \\ 0 & \dfrac{A^2}{2\sigma^2}\displaystyle\sum_{n=0}^{N-1} n^2 & \dfrac{A^2}{2\sigma^2}\displaystyle\sum_{n=0}^{N-1} n \\ 0 & \dfrac{A^2}{2\sigma^2}\displaystyle\sum_{n=0}^{N-1} n & \dfrac{NA^2}{2\sigma^2} \end{bmatrix}$$

Recall… $SNR = \dfrac{A^2}{2\sigma^2}$ and closed form results for these sums

Sinusoid Estimation CRLBs

(using co-factor & det approach… helped by 0's)

Inverting the FIM by hand gives the CRLB matrix… and then extracting the diagonal elements gives the three bounds:

$$\text{var}(\hat{A}) \geq \frac{2\sigma^2}{N} \quad (\text{volts}^2)$$

$$\text{var}(\hat{\Omega}_o) \geq \frac{12}{SNR \times N(N^2-1)} \quad ((\text{rad/sample})^2)$$

$$\text{var}(\hat{\phi}) \geq \frac{2(2N-1)}{SNR \times N(N+1)} \approx \frac{4}{SNR \times N} \quad (\text{rad}^2)$$

To convert to Hz^2 multiply by $(F_s/2\pi)^2$

- **Amp. Accuracy**: **Decreases as 1/*N*, Depends on Noise Variance (not SNR)**
- **Freq. Accuracy**: **Decreases as 1/N^3, Decreases as as 1/S*NR***
- **Phase Accuracy**: **Decreases as 1/*N*, Decreases as as 1/S*NR***

Frequency Estimation CRLBs and Fs

Not in Book

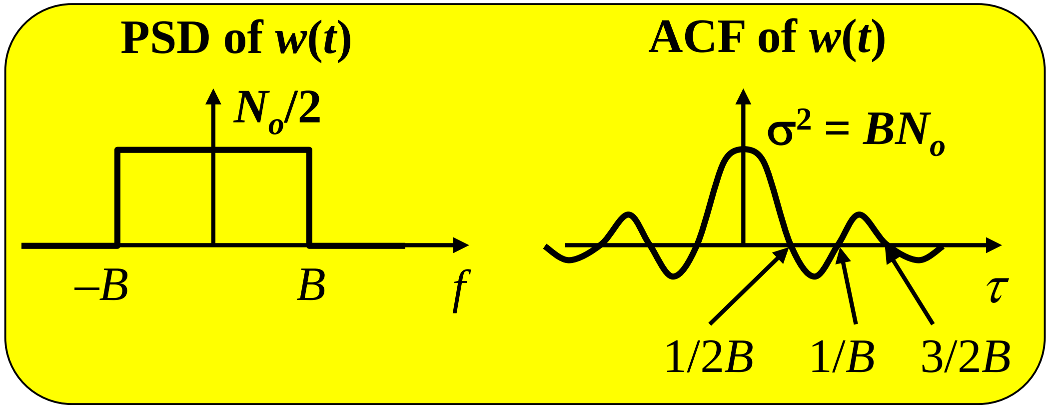
The CRLB for Freq. Est. referred back to the CT is

$$\operatorname{var}(\hat{f}_o) \geq \frac{12 F_s^2}{(2\pi)^2 SNR \times N(N^2-1)} \quad (\text{Hz}^2)$$

Does that mean we do worse if we sample faster than Nyquist?

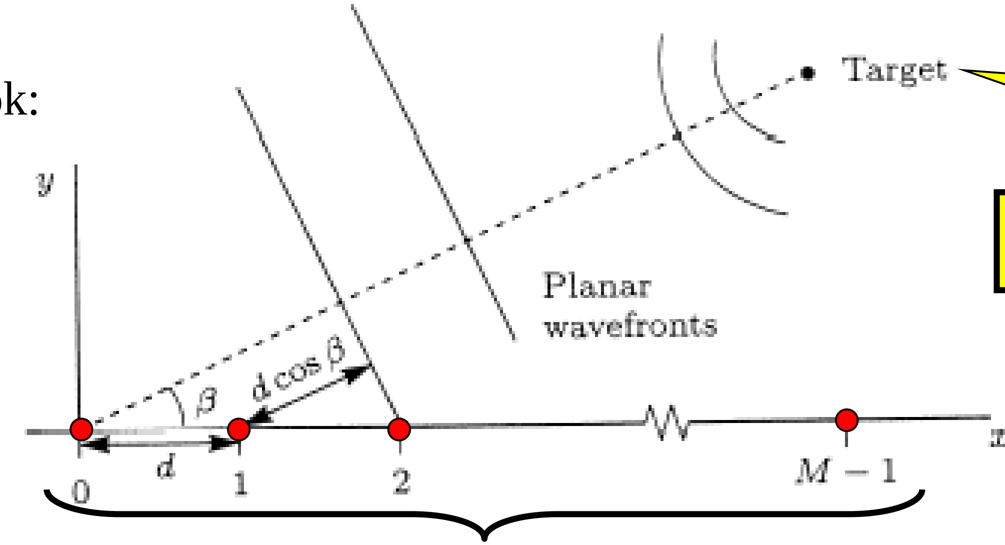
NO!!!!! For a fixed duration T of signal: $N = TF_s$

Also keep in mind that F_s has effect on the noise structure:



Ex. 3 Bearing Estimation CRLB Problem

Figure 3.8
from textbook:



Emits or reflects signal $s(t)$

$$s(t) = A_t \cos(2\pi f_o t + \phi)$$

Simple model

Uniformly spaced linear array with *M* sensors:

- **Sensor Spacing of *d* meters**
- **Bearing angle to target β radians**

Propagation Time to n^{th} Sensor: $t_n = t_0 - n\dfrac{d}{c}\cos\beta \qquad n = 0, 1, \ldots, M-1$

Signal at n^{th} Sensor:

$$\begin{aligned} s_n(t) &= \alpha s(t - t_n) \\ &= A\cos\left(2\pi f_o\left(t - t_0 + n\frac{d}{c}\cos\beta\right) + \phi\right) \end{aligned}$$

Bearing Estimation Snapshot of Sensor Signals

Now instead of sampling each sensor at lots of time instants…
we just grab one "snapshot" of all M sensors at a single instant t_s

$$s_n(t_s) = A\cos\left(2\pi f_o\left(t_s - t_0 + n\frac{d}{c}\cos\beta\right) + \phi\right)$$

$$= A\cos\left(\underbrace{\underbrace{\left(\frac{2\pi f_o}{c}\cos\beta\right)}_{\omega_s} d}_{\Omega_s} n + \tilde{\phi}\right) = A\cos\left(\Omega_s n + \tilde{\phi}\right)$$

Spatial sinusoid w/ spatial frequency Ω_s

Spatial Frequencies:

- ω_s is in rad/meter
- Ω_s is in rad/sensor

For sinusoidal transmitted signal… Bearing Est. reduces to Frequency Est. And… we already know its FIM & CRLB!!!

Bearing Estimation Data and Parameters

Each sample in the snapshot is corrupted by a noise sample…

and these M samples make the data vector $\mathbf{x} = [x[0]\ x[1]\ \ldots\ x[M\text{-}1]\ ]$:

$$x[n] = s_n(t_s) + w[n] = A\cos\left(\Omega_s n + \tilde{\phi}\right) + w[n]$$

Each $w[n]$ is a noise sample that comes from a different sensor so…
Model as uncorrelated Gaussian RVs (same as white temporal noise)
Assume each sensor has same noise variance σ^2

So… the parameters to consider are: $\boldsymbol{\theta} = [A \quad \Omega_s \quad \tilde{\phi}\,]^T$

which get transformed to:

$$\boldsymbol{\alpha} = \mathbf{g}(\boldsymbol{\theta}) = \begin{bmatrix} A \\ \beta \\ \tilde{\phi} \end{bmatrix} = \begin{bmatrix} A \\ \arccos\left(\dfrac{c\Omega_s}{2\pi f_o d}\right) \\ \tilde{\phi} \end{bmatrix}$$

Parameter of interest!

Bearing Estimation CRLB Result

Using the FIM for the sinusoidal parameter problem… together with the transform. of parms result (see book p. 59 for details):

$$\text{var}(\hat{\beta}) \geq \frac{12}{(2\pi)^2 SNR \times M \dfrac{M+1}{M-1}\left(\dfrac{L}{\lambda}\right)^2 \sin^2(\beta)} \quad (\text{rad}^2)$$

Define: $L_r = L/\lambda$ ***Array Length "in wavelengths"***

L = Array physical length in meters
M = Number of array elements
$\lambda = c/f_o$ Wavelength in meters (per cycle)

- **Bearing Accuracy:**
  - **Decreases as 1/*SNR***
  - **Decreases as 1/*M***
  - **Decreases as $1/L_r^2$**
  - **Depends on actual bearing β**
    - **Best at β = π/2 ("Broadside")**
    - **Impossible at β = 0! ("Endfire")**

Low-frequency (i.e., long wavelength) signals need very large physical lengths to achieve good accuracy

Ex. 4 AR Estimation CRLB Problem

Reading Assignment

In speech processing (and other areas) we often model the signal as an AR random process and need to estimate the AR parameters. An AR process has a PSD given by

$$P_{xx}(f;\boldsymbol{\theta}) = \frac{\sigma_u^2}{\left|1 + \sum_{m=1}^{p} a[m]e^{-j2\pi fm}\right|^2}$$

AR Estimation Problem: Given data $x[0]$, $x[1]$, … , $x[N\text{-}1]$ estimate the AR parameter vector

$$\boldsymbol{\theta} = \begin{bmatrix} a[1] & a[2] & \cdots & a[p] & \sigma_u^2 \end{bmatrix}^T$$

This is a hard CRLB to find exactly… but it has been published.

The <u>difficulty</u> comes from the fact that there is <u>no easy direct relationship</u> between the parameters and the data.

It is not a signal plus noise problem

AR Estimation CRLB Asymptotic Approach

Approach: The asymptotic result we discussed is perfect here:

- An AR process is WSS… is required for the Asymp. Result
- Gaussian is often a reasonable assumption… needed for Asymp. Result
- The Asymp. Result is in terms of partial derivatives of the PSD… and that is exactly the form in which the parameters are clearly displayed!

Recall:

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} \approx \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial\left[\ln P_{xx}(f;\boldsymbol{\theta})\right]}{\partial\theta_i}\frac{\partial\left[\ln P_{xx}(f;\boldsymbol{\theta})\right]}{\partial\theta_j}\,df$$

$$\ln P_{xx}(f;\boldsymbol{\theta}) = \ln\frac{\sigma_u^2}{\left|1+\sum_{m=1}^{p}a[m]e^{-j2\pi fm}\right|^2} = \ln\sigma_u^2 - \ln\left|1+\sum_{m=1}^{p}a[m]e^{-j2\pi fm}\right|^2$$

AR Estimation CRLB Asymptotic Result

After taking these derivatives… you get results that can be simplified using properties of FT and convolution.

The final result is:

$$\text{var}(\hat{a}[k]) \geq \frac{\sigma_u^2}{N}\left[\mathbf{R}_{xx}^{-1}\right]_{kk} \qquad k = 1, 2, \ldots, p$$

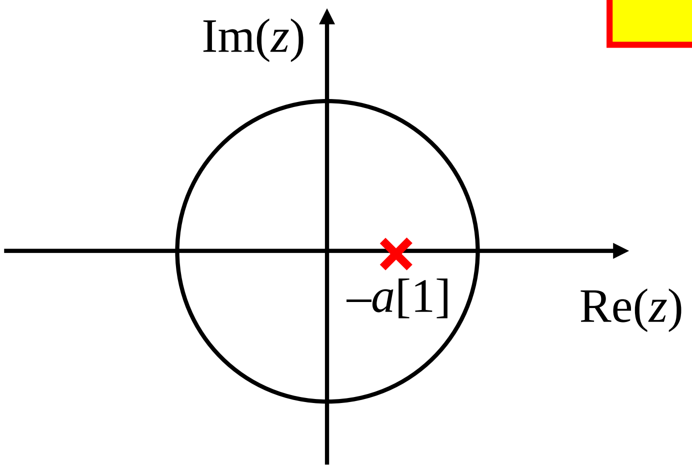
$$\text{var}(\hat{\sigma}_u^2) \geq \frac{2\sigma_u^4}{N}$$

Complicated dependence on AC Matrix!!

Both Decrease as $1/N$

To get a little insight… look at 1st order AR case (p = 1):

$$\text{var}(\hat{a}[1]) \geq \frac{1}{N}(1 - a^2[1])$$



Improves as pole gets closer to unit circle…
PSDs with sharp peaks are easier to estimate