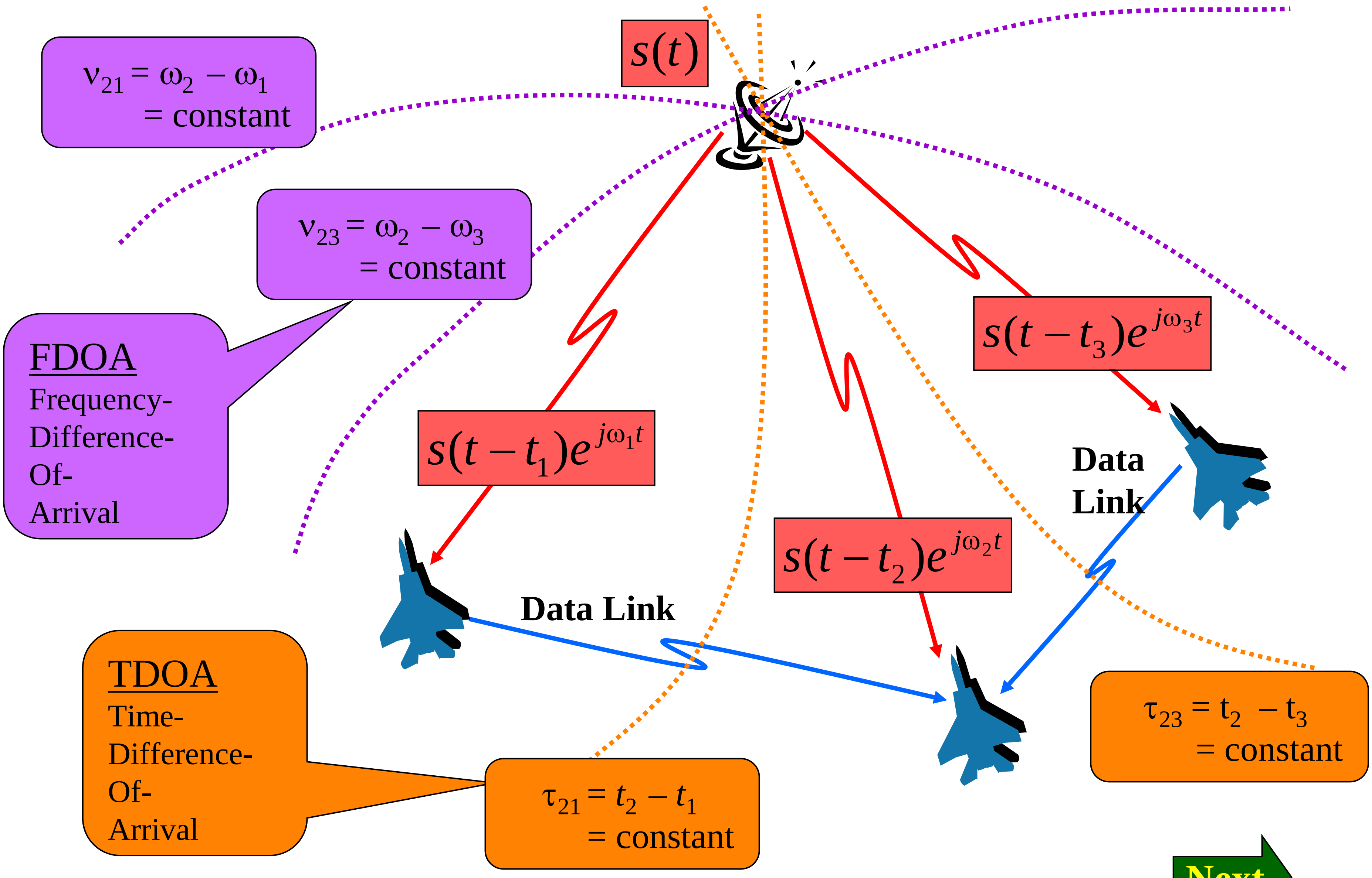
Case Study
TDOA/FDOA Location

- Overview
- Stage 1: Estimating TDOA/FDOA
- Stage 2: Estimating Geo-Location

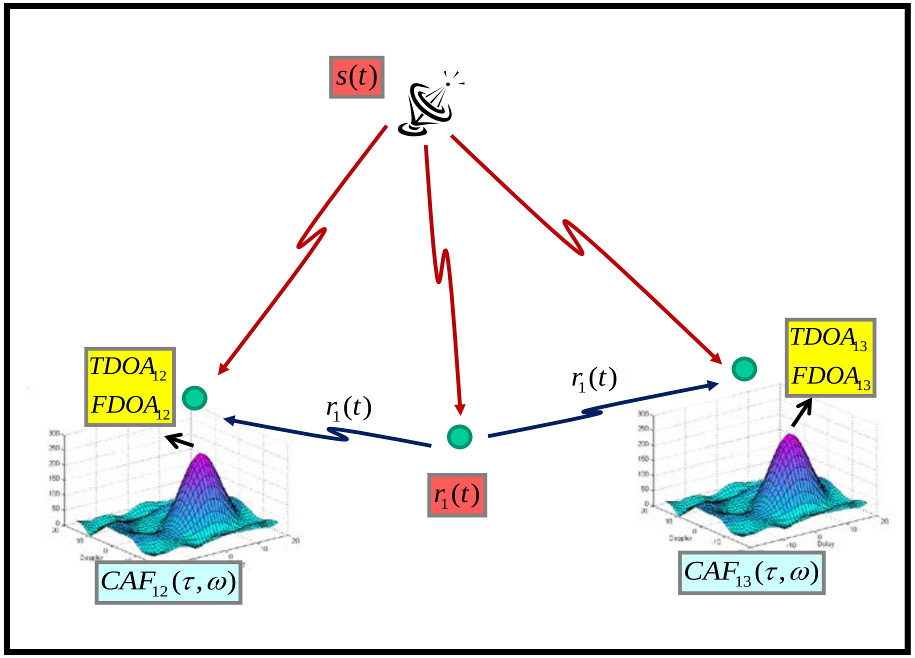
TDOA/FDOA LOCATION

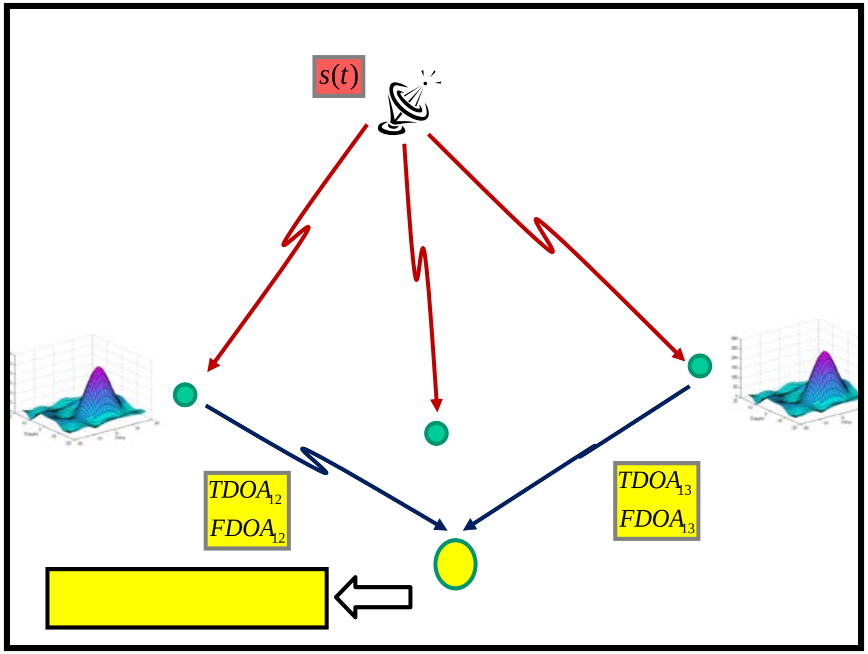


Next

Classical TDOA/FDOA Emitter Location:

- **Stage 1**: Estimate TDOA/FDOA
- **Stage 2**: Estimate the emitter's location from the info from stage 1.





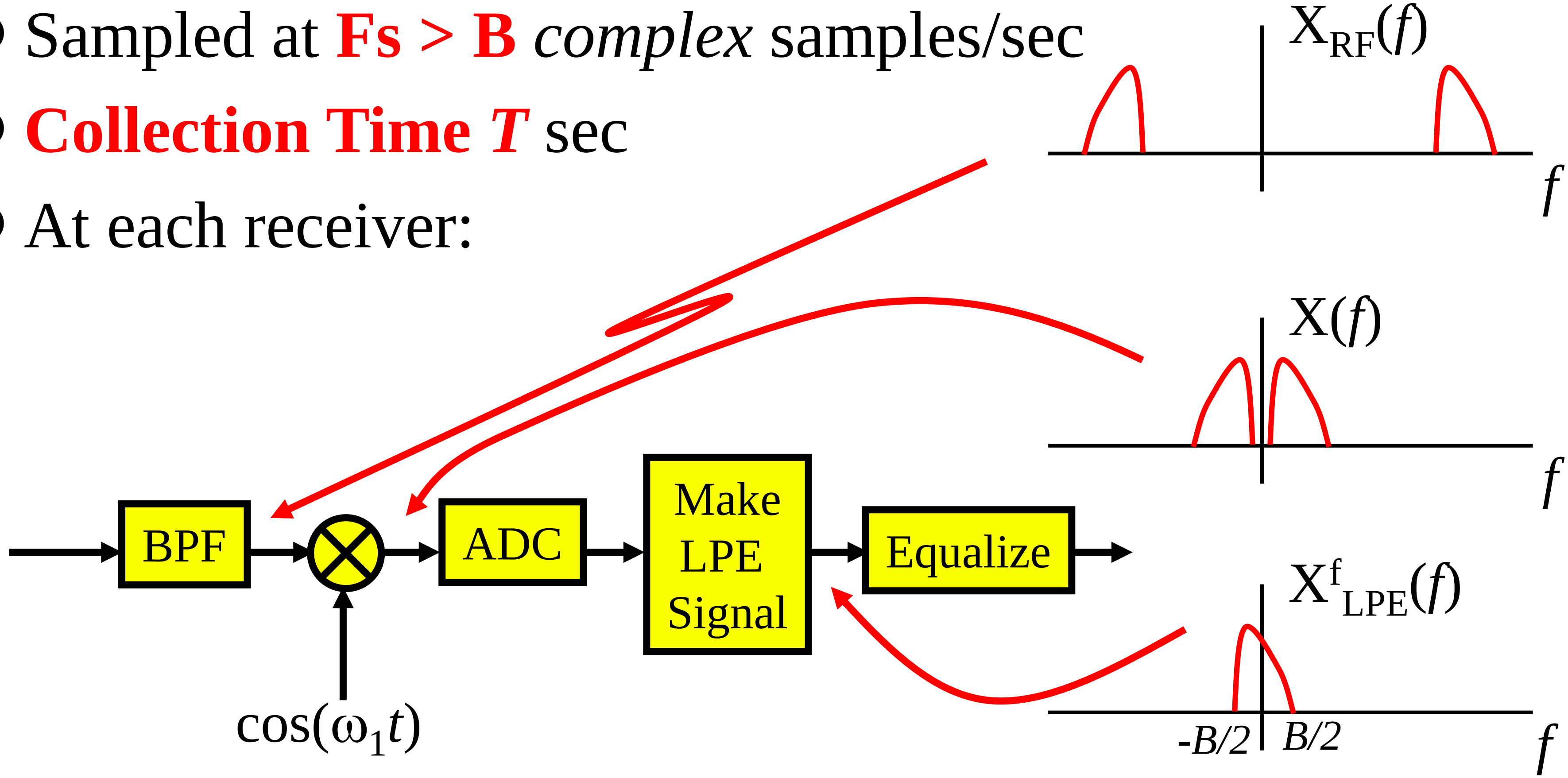
Next

Stage 1: Estimating TDOA/FDOA

Next

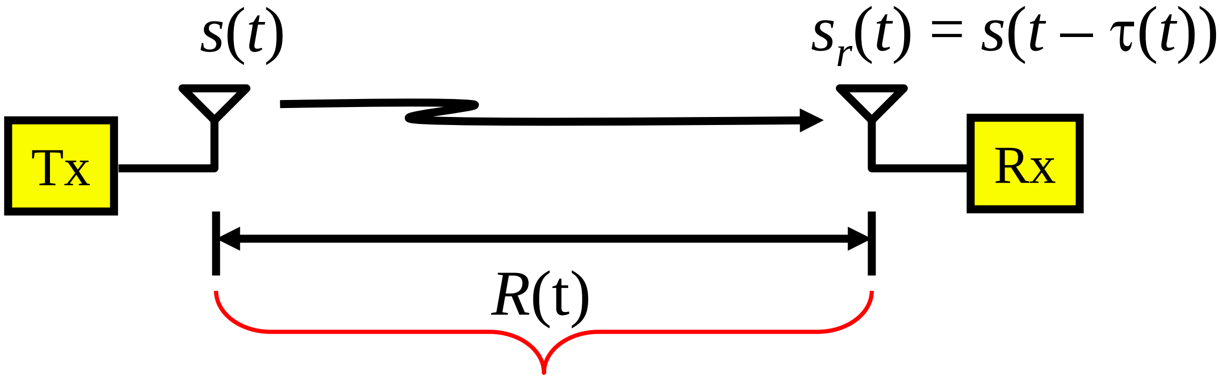
Signal Model

- Will Process Equivalent Lowpass signal, **BW = *B* Hz**
  - Representing RF signal with RF BW = B Hz
- Sampled at **Fs > B** *complex* samples/sec
- **Collection Time *T*** sec
- At each receiver:



Next

DOPPLER & DELAY MODEL



Propagation Time: $\tau(t) = R(t)/c$

$$R(t) = R_o + vt + (a/2)t^2 + \cdots$$

Use linear approximation – assumes small change in velocity over observation interval

For Real BP Signals:

$$s_r(t) = s(t - [R_o + vt]/c) = s([1 - v/c]t - R_o/c)$$

$[1 - v/c]$: Time Scaling

R_o/c: Time Delay: τ_d

Next

DOPPLER & DELAY MODEL (continued)

Analytic Signals Model

$$\tilde{s}(t) = E(t)e^{j[\omega_c t + \phi(t)]}$$

Analytic Signal of Tx

$$\tilde{s}_r(t) = \tilde{s}([1 - v/c]t - \tau_d)$$

$$= E([1 - v/c]t - \tau_d)e^{j\{\omega_c([1-v/c]t-\tau_d)+\phi([1-v/c]t-\tau_d)\}}$$

Analytic Signal of Rx

Now what? Notice that $v << c$ ➔ $(1 - v/c) \approx 1$

Say $v = -300$ m/s (–670 mph) then $v/c = -300/3\text{x}10^8 = -10^{-6}$ ➔ $(1 - v/c) = 1.000001$

Now assume $E(t)$ & $\phi(t)$ vary slowly enough that

$$E([1 - v/c]t) \approx E(t)$$

$$\phi([1 - v/c]t) \approx \phi(t)$$

For the range of v of interest

Called Narrowband Approximation

Next

DOPPLER & DELAY MODEL (continued)

Narrowband Analytic Signal Model

$$\tilde{s}_r(t) = E(t-\tau_d)e^{j\{\omega_c t - \omega_c (v/c)t - \omega_c \tau_d + \phi(t-\tau_d)\}}$$

$$= \underbrace{e^{-j\omega_c \tau_d}}_{\substack{\text{Constant Phase Term} \\ \alpha = -\omega_c \tau_d}} \underbrace{e^{-j\omega_c (v/c)t}}_{\substack{\text{Doppler Shift Term} \\ \omega_d = \omega_c v/c}} \underbrace{e^{j\omega_c t}}_{\text{Carrier Term}} \underbrace{E(t-\tau_d)e^{j\phi(t-\tau_d)}}_{\substack{\text{Transmitted Signal's LPE Signal} \\ \text{Time-Shifted by } \tau_d}}$$

Narrowband Equivalent Lowpass Signal (ELPS) Model

$$\hat{s}_r(t) = e^{j\alpha} e^{-j\omega_d t}\, \hat{s}(t-\tau_d)$$

This is the signal that actually gets processed digitally

Next

Stein's CRLB for TDOA

S. Stein, "Algorithms for Ambiguity Function Processing," *IEEE Trans. on ASSP*, June 1981

Most well-known form is for the C-T version of the problem:

$$\sigma_{TDOA} \geq \frac{1}{2\pi\sqrt{2}\, B_{rms} \sqrt{BT \times SNR_{eff}}}$$ "seconds"

$$B_{rms}^2 = \frac{\int f^2 |S(f)|^2 df}{\int |S(f)|^2 df}$$

$$\sigma_{FDOA} \geq \frac{1}{2\pi\sqrt{2}\, T_{rms} \sqrt{BT \times SNR_{eff}}}$$ "Hz"

$$T_{rms}^2 = \frac{\int t^2 |s(t)|^2 dt}{\int |s(t)|^2 dt}$$

***BT* = Time-Bandwidth Product (≈ *N*, number of samples in DT)**

***B* = Noise Bandwidth of Receiver (Hz)**

***T* = Collection Time (sec)**

Problem with Stein's CRLBs

M. Fowler X. Hu, "Signal Models for TDOA/FDOA Estimation," *IEEE T. AES*, Oct. 2008.

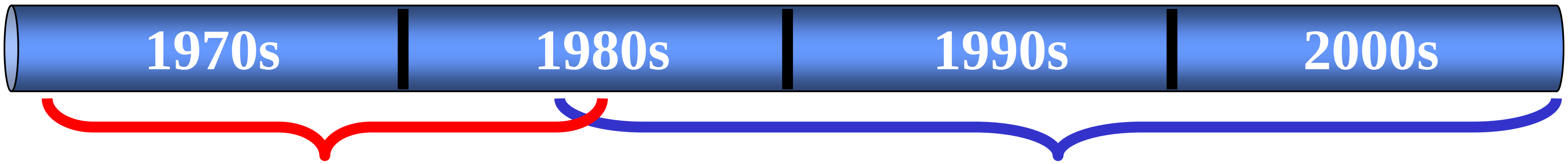
Stein's paper does not derive these CRLB results… rather they are just stated.

There is no mention of what signal model is assumed….

And, it turns out that matters very much!!!

Next

TDOA/FDOA CRLB History Lesson



Sonar-Driven Research

Hann, Tretter, Knapp, Carter, Schultheis, Weinstein, Etc.

Radar/Comm-Driven Research

Stein, Chestnut, Berger, Blahut, Torrieri, Fowler, Yeredor, Etc.

Question: How much of the Sonar TDOA/FDOA estimation work can be carried over to the Radar/Comm arena??

Answer: Not as much as many Radar/Comm researchers/practitioners think!

Next

Signals: Sonar vs. RF

M. Fowler X. Hu, "Signal Models for TDOA/FDOA Estimation," *IEEE T. AES*, Oct. 2008.

Two sampled passively-received complex-valued baseband signals:

$$r_1[n] = e^{j\phi} s(nT - \tau_1) e^{j\nu_1 nT} + w_1[n]$$

$$r_2[n] = s(nT) + w_2[n]$$

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}$$

Noise Model

- Zero-mean WSS processes
- Gaussian
- Independent of each other

This much is the same for each case…

At least when the narrowband approximation can be used… which we assume here so we can focus on the impact of differences in the *statistical* model.

Next

Signal Models: Sonar vs. RF

M. Fowler X. Hu, "Signal Models for TDOA/FDOA Estimation," *IEEE T. AES*, Oct. 2008.

- **Passive Sonar**
  - Signal = Sound from Boat
  - "Erratic" signal behavior
  - Model as Random Process
    - Zero-mean WSS
    - Gaussian
    - Independent of Noise
  - Expected values taken over signal + noise ensemble
    - Estimation accuracy is average over all possible noises <u>and</u> signals

- **Passive Radar/Comm**
  - Signal = Pulse Train
  - Structured signal behavior
  - Model as Deterministic
    - Specific pulse shape
    - Pulse width & spacing
  - Expected values taken over only noise ensemble
    - Estimation accuracy is average over all possible noises <u>for one specific signal</u>

Next

PDFs: Sonar vs. RF

M. Fowler X. Hu, "Signal Models for TDOA/FDOA Estimation," *IEEE T. AES*, Oct. 2008.

- For both cases the received data vector… is Gaussian.
- But how TDOA/FDOA is embedded is very different.

This is the key… it impacts significant differences in:
- **Fisher Info Matrix (FIM) / Cramer-Rao Bound (CRB)**
- **ML Estimator Structure**

Passive Sonar PDF:

TDOA/FDOA in Covariance

$$p_{ac}(\mathbf{r};\boldsymbol{\theta}) = \frac{1}{\det\left(\pi \mathbf{C}_{\mathbf{r}}(\boldsymbol{\theta})\right)} \exp\left\{-\mathbf{r}^H \left(\mathbf{C}_{\mathbf{r}}(\boldsymbol{\theta})\right)^{-1} \mathbf{r}\right\}$$

Passive Radar/Comm PDF

TDOA/FDOA in Mean

$$p_{em}(\mathbf{r};\boldsymbol{\theta}) = \frac{1}{\det\left(\pi \mathbf{C}_{\mathbf{r}}\right)} \exp\left\{-(\mathbf{r}-\mathbf{s}_{\boldsymbol{\theta}})^H \mathbf{C}_{\mathbf{r}}^{-1} (\mathbf{r}-\mathbf{s}_{\boldsymbol{\theta}})\right\}$$

Next

FIM/CRB: Sonar vs. RF

M. Fowler X. Hu, "Signal Models for TDOA/FDOA Estimation," *IEEE T. AES*, Oct. 2008.

- For complex Gaussian case the FIM elements are:

$$[\mathbf{J}_{gg}]_{ij} = 2\,\mathrm{Re}\left(\left[\frac{\partial \boldsymbol{\mu}_{\boldsymbol{\theta}}}{\partial \theta_i}\right]^H \mathbf{C}^{-1}(\boldsymbol{\theta})\left[\frac{\partial \boldsymbol{\mu}_{\boldsymbol{\theta}}}{\partial \theta_j}\right]\right) + \mathrm{tr}\left(\mathbf{C}_{\boldsymbol{\theta}}^{-1}\frac{\partial\, \mathbf{C}_{\boldsymbol{\theta}}}{\partial\, \theta_i}\mathbf{C}_{\boldsymbol{\theta}}^{-1}\frac{\partial\, \mathbf{C}_{\boldsymbol{\theta}}}{\partial\, \theta_j}\right)$$

- Leads to VERY different forms for the two cases:

Passive Sonar FIM:

$$[\mathbf{J}_{sonar}]_{ij} = \mathrm{tr}\left(\mathbf{C}_{\boldsymbol{\theta}}^{-1}\frac{\partial\, \mathbf{C}_{\boldsymbol{\theta}}}{\partial\, \theta_i}\mathbf{C}_{\boldsymbol{\theta}}^{-1}\frac{\partial\, \mathbf{C}_{\boldsymbol{\theta}}}{\partial\, \theta_j}\right)$$

First developed by Bangs.. falls out of general case
Difficult to assess… usually use "Whittles Theorem"
Depends on Noise PSD as well as Signal PSD

Passive Radar/Comm FIM:

$$[\mathbf{J}_{radar}]_{ij} = 2\,\mathrm{Re}\left(\left[\frac{\partial \mathbf{s}_{\boldsymbol{\theta}}}{\partial \theta_i}\right]^H \mathbf{C}^{-1}\left[\frac{\partial \mathbf{s}_{\boldsymbol{\theta}}}{\partial \theta_j}\right]\right)$$

Easy to numerically assess…
Depends on specific signal structure

Next

Correct CRLB for RF Signals

A. Yeredor & E. Angel, "Joint TDOA and FDOA Estimation: A Conditional Bound and Its Use for Optimality Weighted Localization", IEEE T. SP April 2011.

Fowler & Hu ➔ Yeredor & Angel to consider "specific signal" case

"…bounds derived under an assumption of a stochastic source signal are associated with the "average" performance, averaged not only over noise realizations, but also over different source signal realizations, all drawn from the same statistical model."

"It might be of greater interest to obtain a "signal-specific" bound, namely: for a given realization of the source signal, to predict the attainable performance when averaged only over different realizations of the noise. Such a bound can relate more accurately to the specific structure of the specific signal."

Next

Correct CRLB for RF Signals

A. Yeredor & E. Angel

$$\left.\begin{aligned} r_1[n] &= s(nT) + w_1[n] \\ r_2[n] &= ae^{j\phi}\underbrace{s(nT-\tau)}_{\triangleq s_\tau[n]}e^{j\nu nT} + w_2[n] \end{aligned}\right\} \quad -\frac{N}{2} \le n \le \frac{N}{2}-1$$

Signal Model

- Deterministic
- Complex Baseband
- $s[n]$ itself is **UN**-Known
  - Must Estimate!

Noise Model

- Zero-mean WSS processes
- White (can generalize to colored noise)
- Gaussian
- Independent of each other
- Complex Baseband

Define:

$$\mathbf{s} \triangleq \left[s\left[-\frac{N}{2}\right] \quad s\left[-\frac{N}{2}+1\right] \quad \cdots \quad s\left[\frac{N}{2}-1\right] \right]^T$$

$$\mathbf{s}_\tau \triangleq \left[s_\tau\left[-\frac{N}{2}\right] \quad s_\tau\left[-\frac{N}{2}+1\right] \quad \cdots \quad s_\tau\left[\frac{N}{2}-1\right] \right]^T$$

Next

Correct CRLB for RF Signals

A. Yeredor & E. Angel

Now using property of DFT: $\mathbf{s}_\tau = \mathbf{F}^H \mathbf{D}_\tau \mathbf{F}\mathbf{s}$

(Pad zeros to account for DFT circular nature)

$\mathbf{F}$ is (unitary) DFT matrix: $\mathbf{F} = \frac{1}{\sqrt{N}} \exp\left(-j\frac{2\pi}{N} \cdot \mathbf{n}\mathbf{n}^T\right)$

$\mathbf{D}_\tau$ is "delay" matrix: $\mathbf{D}_\tau = diag\left\{\exp\left(-j\frac{2\pi}{N} \cdot \mathbf{n} \cdot \tau\right)\right\}$

$\mathbf{D}_\nu$ is "doppler" matrix: $\mathbf{D}_\nu = diag\left\{\exp\left(-j \cdot \mathbf{n} \cdot \nu\right)\right\}$

$$\mathbf{n} \triangleq \begin{bmatrix} -\frac{N}{2} \\ -\frac{N}{2}+1 \\ \vdots \\ \frac{N}{2}-1 \end{bmatrix}$$

Then get:

$$\mathbf{r}_1 = \mathbf{s} + \mathbf{v}_1$$
$$\mathbf{r}_2 = ae^{j\phi}\mathbf{D}_\nu \left[\mathbf{F}^H \mathbf{D}_\tau \mathbf{F}\right]\mathbf{s} + \mathbf{v}_2$$

Models Doppler — Unit: "rad/sample" $(-\pi, \pi)$

Models Delay — Unit: "samples"

Next

Correct CRLB for RF Signals

A. Yeredor & E. Angel

$$\mathbf{r}_1 = \mathbf{s} + \mathbf{v}_1$$

$$\mathbf{r}_2 = ae^{j\phi}\underbrace{\mathbf{D}_\nu\left[\mathbf{F}^H\mathbf{D}_\tau\mathbf{F}\right]}_{\triangleq \mathbf{Q}_{\tau,\nu}}\mathbf{s} + \mathbf{v}_2$$

Recall: Must treat **s** as **Unknown**!

Parameters to Estimate: $\boldsymbol{\theta} = \left[\mathrm{Re}\{\mathbf{s}\} \quad \mathrm{Im}\{\mathbf{s}\} \quad \underbrace{a \quad \phi \quad \tau \quad \nu}_{\triangleq \boldsymbol{\gamma}}\right]$

Data Vector (Gaussian): $\mathbf{r} = \left[\mathbf{r}_1^T \quad \mathbf{r}_2^T\right]^T$

$$\boldsymbol{\mu}_{\boldsymbol{\theta}} \triangleq E\{\mathbf{r}\} = \begin{bmatrix} \mathbf{s} \\ ae^{j\phi}\mathbf{D}_\nu\left[\mathbf{F}^H\mathbf{D}_\tau\mathbf{F}\right]\mathbf{s} \end{bmatrix}$$

$$\mathbf{C}_{\boldsymbol{\theta}} = \mathrm{cov}\{\mathbf{r}\} \triangleq \boldsymbol{\Lambda} = \begin{bmatrix} \sigma_1^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sigma_2^2\mathbf{I} \end{bmatrix}$$

No $\boldsymbol{\theta}$ dependence!

General Gaussian FIM elements:

$$[\mathbf{J}_{\boldsymbol{\theta}}]_{ij} = 2\,\mathrm{Re}\left(\left[\frac{\partial \boldsymbol{\mu}_{\boldsymbol{\theta}}}{\partial \theta_i}\right]^H \mathbf{C}_{\boldsymbol{\theta}}^{-1}\left[\frac{\partial \boldsymbol{\mu}_{\boldsymbol{\theta}}}{\partial \theta_j}\right]\right) + \mathrm{tr}\left(\mathbf{C}_{\boldsymbol{\theta}}^{-1}\frac{\partial \mathbf{C}_{\boldsymbol{\theta}}}{\partial \theta_i}\mathbf{C}_{\boldsymbol{\theta}}^{-1}\frac{\partial \mathbf{C}_{\boldsymbol{\theta}}}{\partial \theta_j}\right)$$

Easy Inversion!

This term is zero!

Next

Correct CRLB for RF Signals

A. Yeredor & E. Angel

$$\mathbf{r}_1 = \mathbf{s} + \mathbf{v}_1$$

$$\mathbf{r}_2 = a e^{j\phi} \underbrace{\mathbf{D}_\nu \left[\mathbf{F}^H \mathbf{D}_\tau \mathbf{F} \right]}_{\triangleq \mathbf{Q}_{\tau,\nu}} \mathbf{s} + \mathbf{v}_2$$

$$\mathbf{J}_\theta = 2\,\mathrm{Re}\left(\left[\frac{\partial \boldsymbol{\mu}_\theta}{\partial \boldsymbol{\theta}} \right]^H \boldsymbol{\Lambda}^{-1} \left[\frac{\partial \boldsymbol{\mu}_\theta}{\partial \boldsymbol{\theta}} \right] \right)$$

$$\boldsymbol{\theta} = \left[\mathrm{Re}\{\mathbf{s}\} \quad \mathrm{Im}\{\mathbf{s}\} \quad \underbrace{a \quad \phi \quad \tau \quad \nu}_{\triangleq \boldsymbol{\gamma}} \right]$$

$$\mathbf{J}_\theta = \frac{2}{\sigma_1^2} \begin{bmatrix} \left(1+\eta a^2\right)\mathbf{I} & \mathbf{0} & \eta a\,\mathrm{Re}\{\mathbf{B}\} \\ \mathbf{0} & \left(1+\eta a^2\right)\mathbf{I} & \eta a\,\mathrm{Im}\{\mathbf{B}\} \\ \eta a\,\mathrm{Re}\{\mathbf{B}^H\} & -\eta a\,\mathrm{Im}\{\mathbf{B}^H\} & \eta\,\mathrm{Re}\{\mathbf{G}^H\mathbf{G}\} \end{bmatrix}$$

$$\eta \triangleq \frac{\sigma_1^2}{\sigma_2^2} \qquad \mathbf{G} \triangleq \frac{\partial a e^{j\phi} \mathbf{Q}_{\tau,\nu} \mathbf{s}}{\partial \boldsymbol{\gamma}} \qquad \mathbf{B} \triangleq e^{-j\phi} Q_{\tau,\nu}^H \mathbf{G} \qquad \boldsymbol{\gamma} = [a \quad \phi \quad \tau \quad \nu]$$

Now could get the CRLB matrix for full parameter vector:

$$\mathbf{CRLB}_\theta = \mathbf{J}_\theta^{-1}$$

But we really only want w.r.t. $\boldsymbol{\gamma}$

Next

Correct CRLB for RF Signals

A. Yeredor & E. Angel

Define: $$\mathbf{CRLB}_{\boldsymbol{\theta}} = \begin{bmatrix} \mathbf{J}^{-1}_{\mathrm{Re}\{\mathbf{s}\},\mathrm{Im}\{\mathbf{s}\}} & ?? \\ ?? & \mathbf{J}^{-1}_{\boldsymbol{\gamma}} \end{bmatrix} \Rightarrow \mathbf{CRLB}_{\boldsymbol{\gamma}} = \mathbf{J}^{-1}_{\boldsymbol{\gamma}}$$

Then.. use a "math trick" called "Schur Complement" we get

$$\mathbf{J}_{\boldsymbol{\gamma}} = \frac{2}{a^2\sigma_1^2 + \sigma_2^2}\mathrm{Re}\left[\mathbf{G}^H\mathbf{G}\right]$$

Evaluating the elements in $\mathbf{G}^H\mathbf{G}$ leads to noting this form:

$$\mathbf{J}_{\boldsymbol{\gamma}} = \begin{bmatrix} J_a & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{\phi,\tau,\nu} \end{bmatrix}$$

Amplitude parameters virtually always decouple like this!

So… really only need this!

Next

Correct CRLB for RF Signals

A. Yeredor & E. Angel

$$\mathbf{r}_1 = \mathbf{s} + \mathbf{v}_1$$

$$\mathbf{r}_2 = ae^{j\phi} \underbrace{\mathbf{D}_\nu \left[\mathbf{F}^H \mathbf{D}_\tau \mathbf{F}\right]}_{\triangleq \mathbf{Q}_{\tau,\nu}} \mathbf{s} + \mathbf{v}_2$$

The final result for the FIM of interest is:

$$\mathbf{J}_{\phi,\tau,\nu} = \begin{bmatrix} \mathbf{s}^H\mathbf{s} & -\mathbf{s}^H\mathbf{s}' & \tilde{\mathbf{s}}^H\mathbf{N}\tilde{\mathbf{s}} \\ -\mathbf{s}^H\mathbf{s}' & \mathbf{s}'^H\mathbf{s}' & -\mathrm{Re}\left\{\mathbf{s}'^H\mathbf{Q}_{\tau,\nu}^H\mathbf{N}\tilde{\mathbf{s}}\right\} \\ \tilde{\mathbf{s}}^H\mathbf{N}\tilde{\mathbf{s}} & -\mathrm{Re}\left\{\mathbf{s}'^H\mathbf{Q}_{\tau,\nu}^H\mathbf{N}\tilde{\mathbf{s}}\right\} & \tilde{\mathbf{s}}^H\mathbf{N}^2\tilde{\mathbf{s}} \end{bmatrix}$$

$$\tilde{\mathbf{s}} = \mathbf{Q}_{\tau,\nu}\mathbf{s} = \mathbf{D}_\nu\left[\mathbf{F}^H\mathbf{D}_\tau\mathbf{F}\right]\mathbf{s}$$

$$\mathbf{s}' = \frac{2\pi}{N}\mathbf{F}^H\mathbf{N}\mathbf{F}\mathbf{s}$$

$$\mathbf{F} = \frac{1}{\sqrt{N}}\exp\left(-j\frac{2\pi}{N}\cdot\mathbf{n}\mathbf{n}^T\right)$$

$$\mathbf{D}_\tau = diag\left\{\exp\left(-j\frac{2\pi}{N}\cdot\mathbf{n}\cdot\tau\right)\right\}$$

$$\mathbf{D}_\nu = diag\left\{\exp\left(-j\cdot\mathbf{n}\cdot\nu\right)\right\}$$

$$\mathbf{N} = diag\left\{\mathbf{n}\right\}$$

$$\mathbf{n} \triangleq \begin{bmatrix} -\frac{N}{2} \\ -\frac{N}{2}+1 \\ \vdots \\ \frac{N}{2}-1 \end{bmatrix}$$

So… use all these boxes to compute this J then invert it to get the CRLB!

Next

Correct CRLB for RF Signals

A. Yeredor & E. Angel

We can interpret some of these FIM terms:

$$\mathbf{J}_{\phi,\tau,\nu} = \begin{bmatrix} \mathbf{s}^H\mathbf{s} & -\mathbf{s}^H\mathbf{s}' & \tilde{\mathbf{s}}^H\mathbf{N}\tilde{\mathbf{s}} \\ -\mathbf{s}^H\mathbf{s}' & \mathbf{s}'^H\mathbf{s}' & -\mathrm{Re}\left\{\mathbf{s}'^H\mathbf{Q}_{\tau,\nu}^H\mathbf{N}\tilde{\mathbf{s}}\right\} \\ \tilde{\mathbf{s}}^H\mathbf{N}\tilde{\mathbf{s}} & -\mathrm{Re}\left\{\mathbf{s}'^H\mathbf{Q}_{\tau,\nu}^H\mathbf{N}\tilde{\mathbf{s}}\right\} & \tilde{\mathbf{s}}^H\mathbf{N}^2\tilde{\mathbf{s}} \end{bmatrix}$$

$$\mathbf{s}^H\mathbf{s} = \sum_n s^2[n]$$

$$\mathbf{s}'^H\mathbf{s}' = \left(\frac{2\pi}{N}\right)^2 (\mathbf{F}\mathbf{s})^H \mathbf{N}^2 (\mathbf{F}\mathbf{s})$$

$$\tilde{\mathbf{s}}^H\mathbf{N}^2\tilde{\mathbf{s}} \approx \mathbf{s}^H\mathbf{N}^2\mathbf{s}$$

Energy **Like RMS BW Term** **Like RMS Duration Term**

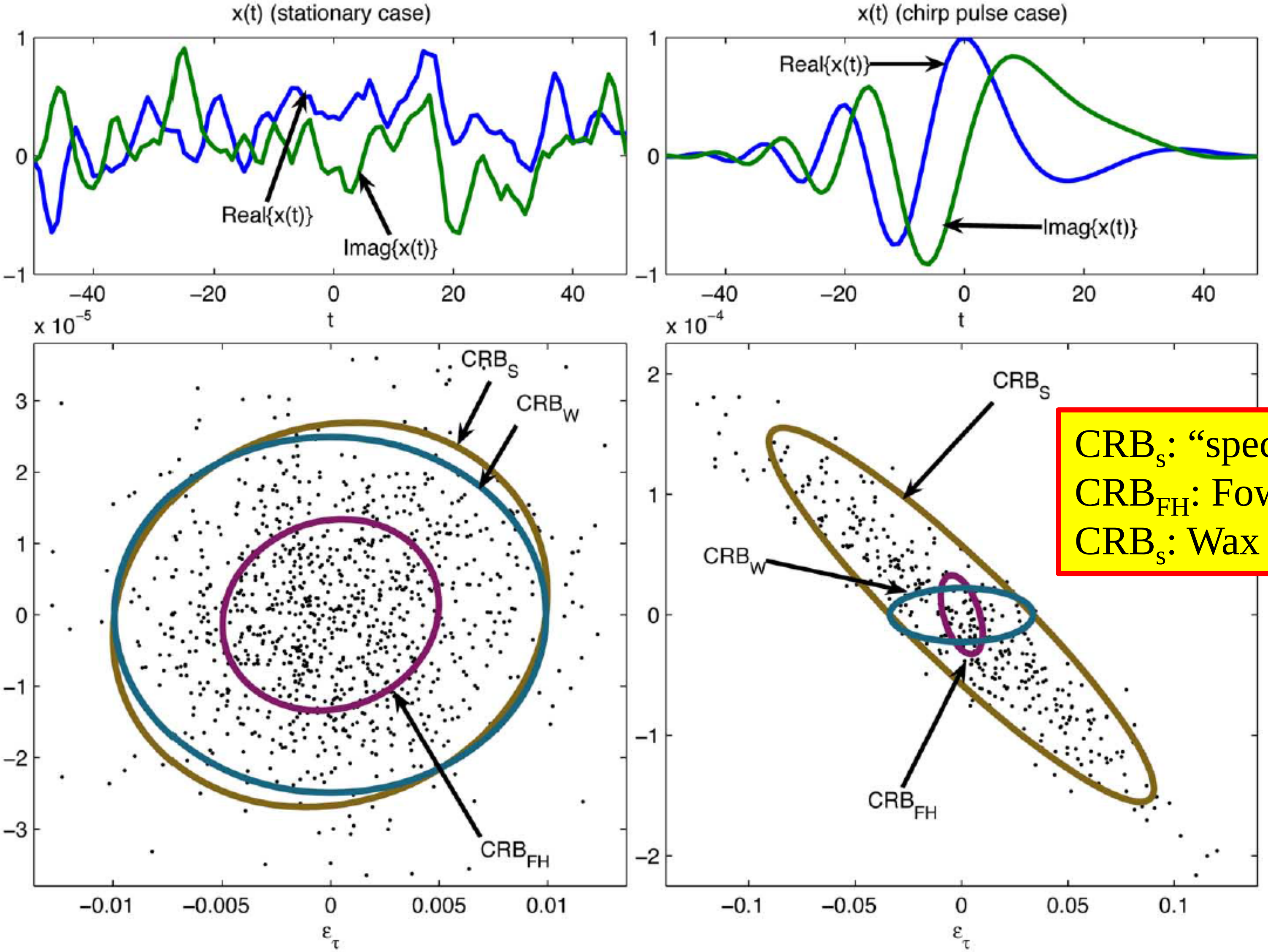
$$B_{rms}^2 = \frac{\int f^2 |S(f)|^2 \, df}{\int |S(f)|^2 \, df}$$

$$T_{rms}^2 = \frac{\int t^2 |s(t)|^2 \, dt}{\int |s(t)|^2 \, dt}$$

Next

Correct CRLB for RF Signals

A. Yeredor & E. Angel



CRB_s: “specific”
CRB_{FH}: Fowler-Hu
CRB_s: Wax for WSS Gaussian

Next

MLE: Sonar vs. Radar/Comm

M. Fowler X. Hu, “Signal Models for TDOA/FDOA Estimation,” *IEEE T. AES*, Oct. 2008.

- For *general* Gaussian case set $\partial \ln\left\{p_{gg}(\mathbf{r};\boldsymbol{\theta})\right\}/\partial\theta_i = 0$

$$\underbrace{-\operatorname{tr}\left(\mathbf{C}_{\boldsymbol{\theta}}^{-1}\frac{\partial \mathbf{C}_{\boldsymbol{\theta}}}{\partial\theta_i}\right) + \left[\mathbf{r}-\boldsymbol{\mu}_{\boldsymbol{\theta}}\right]^H \mathbf{C}_{\boldsymbol{\theta}}^{-1}\frac{\partial \mathbf{C}_{\boldsymbol{\theta}}}{\partial\theta_i}\mathbf{C}_{\boldsymbol{\theta}}^{-1}\left[\mathbf{r}-\boldsymbol{\mu}_{\boldsymbol{\theta}}\right]}_{\text{Covariance Sensitivity}} + \underbrace{2\operatorname{Re}\left\{\left[\mathbf{r}-\boldsymbol{\mu}_{\boldsymbol{\theta}}\right]^H \mathbf{C}_{\boldsymbol{\theta}}^{-1}\frac{\partial \boldsymbol{\mu}_{\boldsymbol{\theta}}}{\partial\theta_i}\right\}}_{\text{Mean Sensitivity}} = 0$$

Covariance Sensitivity | **Mean Sensitivity**

- **Passive Sonar**
  - Derived by Weinstein, Wax
  - Showed trace term = 0

- **Passive Radar/Comm**
  - Derived by Stein

$$\mathbf{r}^H\mathbf{C}_{\boldsymbol{\theta}}^{-1}\frac{\partial \mathbf{C}_{\boldsymbol{\theta}}}{\partial\theta_i}\mathbf{C}_{\boldsymbol{\theta}}^{-1}\mathbf{r} = 0$$

$$\Rightarrow \quad \hat{\boldsymbol{\theta}}_{ML,ac} = \arg\max_{\boldsymbol{\theta}}\left\{-\mathbf{r}^H\mathbf{C}_{\boldsymbol{\theta}}^{-1}\mathbf{r}\right\}$$

$$2\operatorname{Re}\left\{\left[\mathbf{r}-\mathbf{s}_{\boldsymbol{\theta}}\right]^H \mathbf{C}^{-1}\frac{\partial \mathbf{s}_{\boldsymbol{\theta}}}{\partial\theta_i}\right\} = 0$$

$$\Rightarrow \quad \hat{\boldsymbol{\theta}}_{ML,em} = \arg\max_{\boldsymbol{\theta}}\left\{2\operatorname{Re}\left\{\mathbf{r}^H\mathbf{C}^{-1}\mathbf{s}_{\boldsymbol{\theta}}\right\} - \mathbf{s}_{\boldsymbol{\theta}}^H\mathbf{C}^{-1}\mathbf{s}_{\boldsymbol{\theta}}\right\}$$

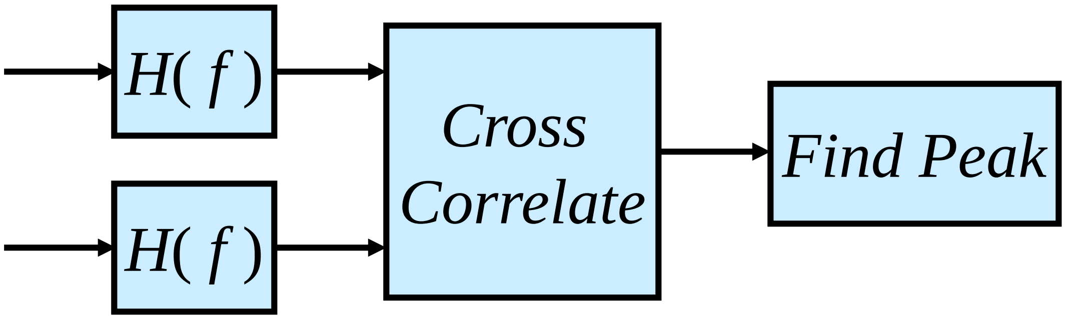
<u>Seem</u> Very Different… but not as much as you’d think

Next

MLE: Sonar vs. Radar/Comm

M. Fowler X. Hu, "Signal Models for TDOA/FDOA Estimation," *IEEE T. AES*, Oct. 2008.

- **Both** lead to cross-correlation with pre-filtering



- Passive Sonar
  - Pre-Filters depend on interplay between Noise PSD & Signal PSD
  - Becomes Std Cross-Correlator when Noise and Signal are white

- Passive Radar/Comm
  - Pre-Filters depend only on Noise PSD, not on signal structure
  - Becomes Std Cross-Correlator when Noise is white… regardless of signal

Next

ML Estimator for TDOA/FDOA

S. Stein, "Differential Delay/Doppler ML Estimation with Unknown Signals," *IEEE Trans. on SP*, 1993.

Two received CT signals in ELPS form (complex) observed over $(0,T)$:

$$y_1(t) = x(t) + n_1(t) \qquad (1a)$$

$x(t)$ itself is unknown!

$$y_2(t) = \alpha x(t+\tau) \exp[j2\pi\nu(t+\tau)] + n_2(t). \qquad (1b)$$

Parameters: Complex Amplitude (α), Delay (τ), Doppler (ν), Delay (τ)

The signal $x(t)$ has bandwidth of B Hz
The time-BW product is assumed large: $BT \gg 1$

Next

S. Stein, "Differential Delay/Doppler ML Estimation with Unknown Signals," *IEEE Trans. on SP*, 1993.

$BT >> 1$ yileds a common trick: analysis in freq domain is easier.

CTFT View:

$$Y_1(f) = X(f) + N_1(f) \quad (2a)$$

$$Y_2(f) = \alpha X(f - \nu) \exp(j2\pi f\tau) + N_2(f). \quad (2b)$$

Now convert this into a DFT view for the DT problem

DFT View:

Recall: X unknown!

$$Y_{1m} = X_m + N_{1m} \quad (3a)$$

$$Y_{2m} = \alpha X_{m-F} W^m + N_{2m} \quad (3b)$$

Doppler

Δf is DFT spacing

$$W = \exp(j2\pi\tau\Delta f). \quad (3c)$$

Assume the Doppler shift is an integer multiple (F) of the DFT spacing

Benefit of converting to Frequency Domain:
$\{N_{im}\}$, $m = 0, 1, \ldots, N$-1 are independent RVs!
(Even when noise is correlated in time!!!)

Covariances: $\mathbf{C}_i \triangleq \text{diag}\left\{P_{i0}, P_{i1}, \ldots, P_{iN-1}\right\}, \quad i = 1, 2$

Next

S. Stein, "Differential Delay/Doppler ML Estimation with Unknown Signals," *IEEE Trans. on SP*, 1993.

Because of the independence (due to the DFT trick) it is easy to write the PDF of the two observed signals' vectors

$$p(Y_1, Y_2 \,|\, X, \tau, \nu, \alpha) = C \exp\left(-L_1/2\right) \qquad \text{(4a)}$$

Minimize!

where:

$$C = \frac{1}{\sqrt{2\pi |P_1| |P_2|}} \qquad P_i = \prod_m P_{im} \qquad i = 1, 2 \qquad \text{(4b)}$$

$$L_1 = \sum_m \left[\frac{|Y_{1m} - X_m|^2}{P_{1m}} + \frac{|Y_{2m} - \alpha X_{m-F} W^m|^2}{P_{2m}} \right] \qquad \text{(4c)}$$

Re-write L_1:

$$L_1 = \sum_m \left[\frac{|Y_{1m}|^2}{P_{1m}} + \frac{|Y_{2,m+F}|^2}{P_{2,m+F}} \right]$$

No Parm or Signal in here!

$$+ \sum_m \frac{1}{P_m} \left| X_m - P_m \left[\frac{Y_{im}}{P_{1m}} + \alpha^*(W^*)^{m+F} \frac{Y_{2,m+F}}{P_{2,m+F}} \right] \right|^2$$

X only in here!

$$+ \sum_m P_m \left| \frac{Y_{1m}}{P_{1m}} + \alpha^*(W^*)^{m+F} \frac{Y_{2,m+F}}{P_{2,m+F}} \right|^2$$

Other Parms only in here!

where:

$$\frac{1}{P_m} = \frac{1}{P_{1m}} + \frac{|\alpha|^2}{P_{2,m+F}}$$

Next

Remember we need to estimate the signal DFT X too!
It only shows up in second term in L_1:

$$\sum_m \frac{1}{P_m}\left| X_m - P_m \left[\frac{Y_{1m}}{P_{1m}} + \alpha^*(W^*)^{m+F} \frac{Y_{2,m+F}}{P_{2,m+F}} \right] \right|^2$$

This is minimized (to 0) by choosing the signal estimate to be

$$\hat{X}_m = P_m \left[\frac{Y_{1m}}{P_{1m}} + \alpha^*(W^*)^{m+F} \frac{Y_{2,m+F}}{P_{2,m+F}} \right]. \quad (8)$$

"Undo" Doppler and delay to align!

When we plug into (8) the ML estimates for delay, Doppler, amplitude we get a signal estimate

So…

- First term of L_1 is not needed
- Second term of L_1 led to signal estimate
- Third term of L_1 … look at now!

Next

S. Stein, "Differential Delay/Doppler ML Estimation with Unknown Signals," *IEEE Trans. on SP*, 1993.

Third term of L_1 … look at now!

$$\sum_m P_m \left| \frac{Y_{1m}}{P_{1m}} + \alpha^*(W^*)^{m+F} \frac{Y_{2,m+F}}{P_{2.m+F}} \right|^2.$$

$$= \sum_m \left\{ \left[\frac{|N_{1m}|^2}{P_{1m}} + |\alpha|^2 \frac{|Y_{2,m+F}|^2}{P_{2,m+F}} \right] \Big/ \left[\frac{1}{P_{1m}} + \frac{|\alpha|^2}{P_{2,m+F}} \right] \right\}$$

where:

$$= \sum_m \frac{(W^*)^{m+F} Y_{2,m+F} Y_{1m}^*}{P_{2,m+F} + |\alpha|^2 P_{1m}}.$$

It follows that we need to maximize $|K|$… equivalent to maxmizing

$$= \frac{1}{T} \left| \int \frac{\exp(-j2\pi f\tau) Y_1^*(f-\nu) Y_2(f)}{P_2(f) + |\alpha|^2 P_1(f-\nu)} \, df \right|.$$

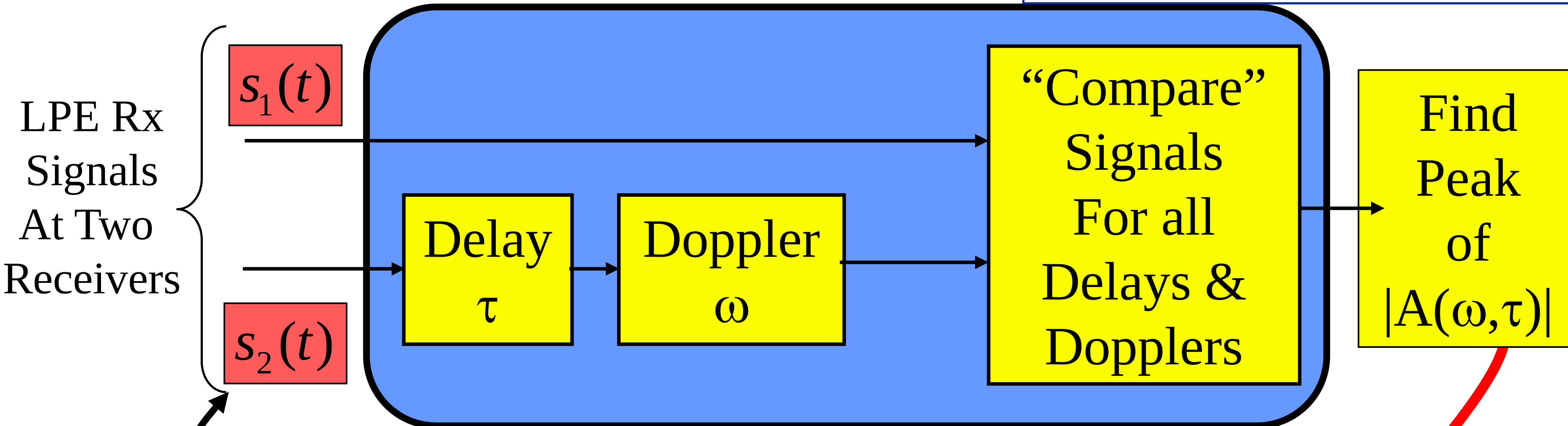
For white noise the denominator is constant and this becomes

$$R(\tau, \nu) = \left| \frac{1}{T} \int \exp(-j2\pi f\tau) Y_1^*(f-\nu) Y_2(f) \, df \right|.$$

Next

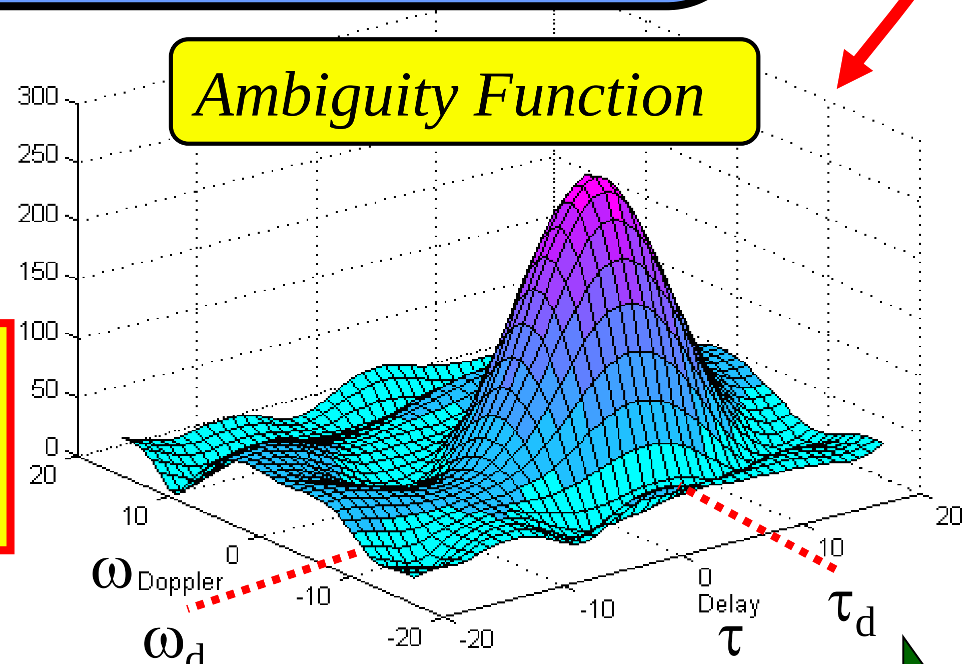
ML Estimator for TDOA/FDOA

S. Stein, "Differential Delay/Doppler ML Estimation with Unknown Signals," *IEEE Trans. on SP*, 1993.



$$= e^{j\alpha} e^{j\omega_d t} s_1(t - \tau_d)$$

$$A(\omega,\tau) = \int_0^T s_1(t)\overline{s_2(t+\tau)}e^{-j\omega t}\,dt$$



Next

COMPUTING THE AMBIGUITY FUNCTION

Direct computation based on the equation for the ambiguity function leads to computationally inefficient methods.

In Prof. Fowler's EECE 521 notes it is shown how to use decimation to efficiently compute the ambiguity function

Next

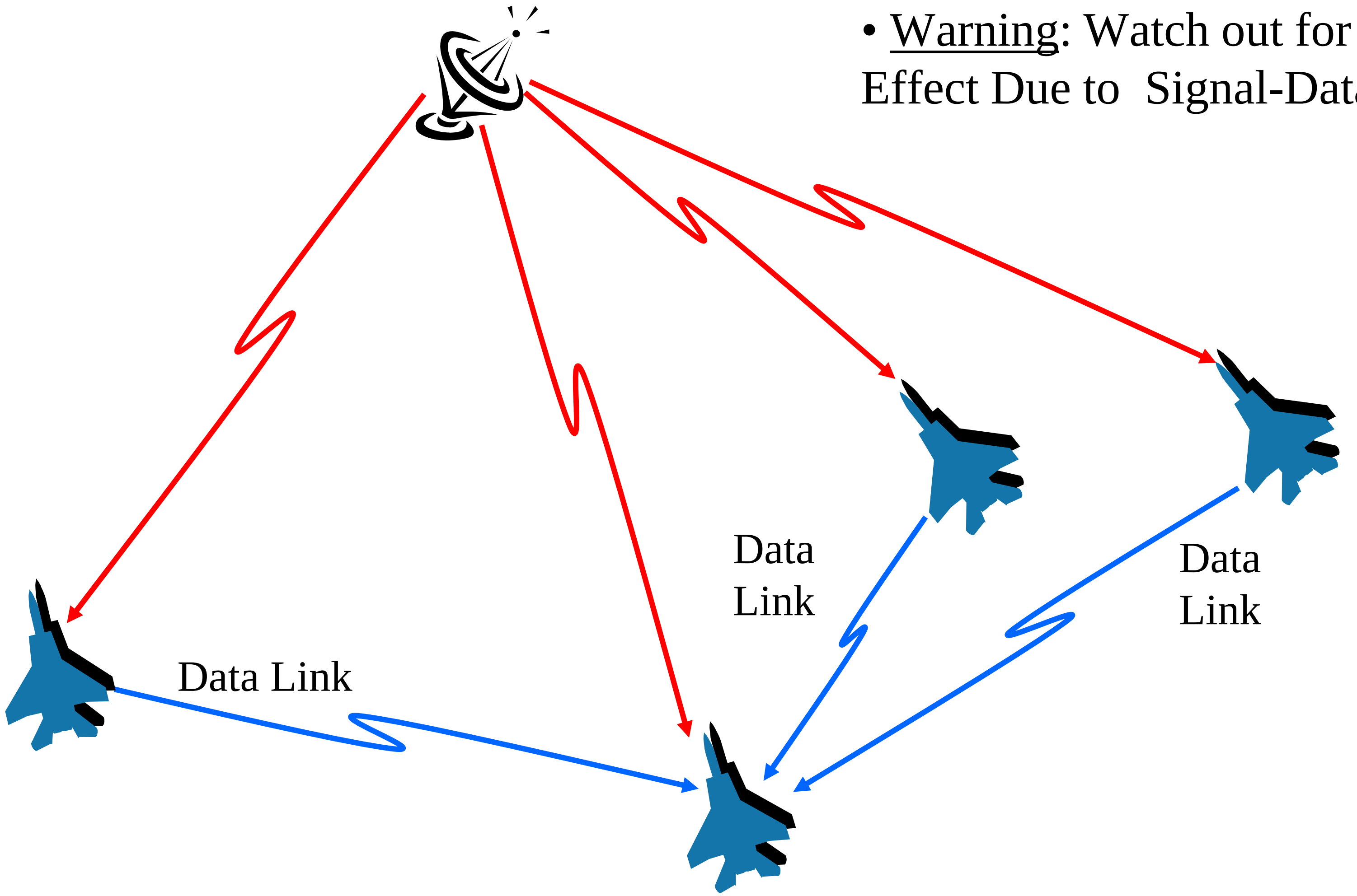
Stage 2: Estimating Geo-Location

Next

TDOA/FDOA LOCATION

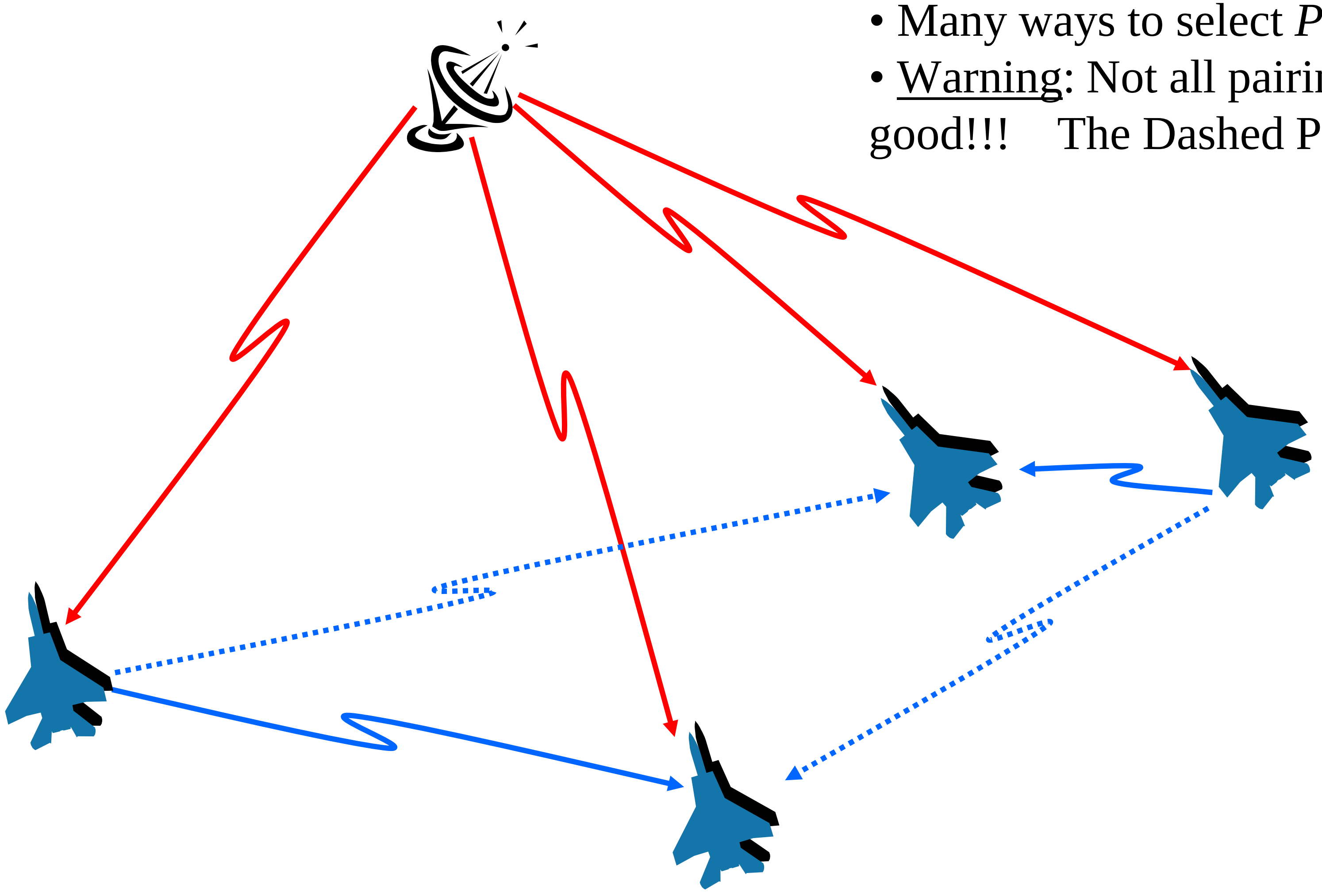
Centralized Network of *P*

• "*P*-Choose-2" Pairs
➢ "*P*-Choose-2" TDOA Measurements
➢ "*P*-Choose-2" FDOA Measurements
• Warning: Watch out for Correlation Effect Due to Signal-Data-In-Common



Next

TDOA/FDOA LOCATION



Pair-Wise Network of *P*

- *P*/2 Pairs
  - ➢ *P*/2 TDOA Measurements
  - ➢ *P*/2 FDOA Measurements
- Many ways to select *P*/2 pairs
- Warning: Not all pairings are equally good!!! The Dashed Pairs are Better

Next

TDOA/FDOA Measurement Model

Given N TDOA/FDOA measurements with corresponding 2×2 Cov. Matrices

$$(\hat{\tau}_1, \hat{\nu}_1), (\hat{\tau}_2, \hat{\nu}_2), \; \ldots \; , (\hat{\tau}_N, \hat{\nu}_N)$$

$$\mathbf{C}_1, \mathbf{C}_2, \; \ldots \; , \mathbf{C}_N$$

Assume pair-wise network, so… TDOA/FDOA pairs are uncorrelated

For notational purposes… define the $2N$ measurements $r(n)$ n = 1, 2, …, $2N$

$$r_{2n-1} = \hat{\tau}_n, \quad n = 1, 2, \ldots, N$$

$$r_{2n} = \hat{\nu}_n, \quad n = 1, 2, \ldots, N$$

$$\underbrace{\mathbf{r} = [r_1 \quad r_2 \quad \cdots \quad r_{2N}]^T}_{\text{Data Vector}}$$

Now, those are the TDOA/FDOA estimates… so the true values are notated as:

$$(\tau_1, \nu_1), (\tau_2, \nu_2), \; \ldots \; , (\tau_N, \nu_N)$$

$$s_{2n-1} = \tau_n, \quad n = 1, 2, \ldots, N$$

$$s_{2n} = \nu_n, \quad n = 1, 2, \ldots, N$$

$$\underbrace{\mathbf{s} = [s_1 \quad s_2 \quad \cdots \quad s_{2N}]^T}_{\text{“Signal” Vector}}$$

Next

TDOA/FDOA Measurement Model (cont.)

Each of these measurements $r(n)$ has an error ε(n) associated with it, so…

$$\mathbf{r} = \mathbf{s} + \boldsymbol{\varepsilon}$$

Because these measurements were estimated using an ML estimator (with sufficiently large number of signal samples) we know that error vector ε is a zero-mean Gaussian vector with cov. matrix C given by:

$$\mathbf{C} = \mathrm{diag}\{\mathbf{C}_1, \mathbf{C}_2, \ \ldots \ , \mathbf{C}_N\} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_N \end{bmatrix}$$

Assumes that TDOA/FDOA pairs are uncorrelated!!!

The true TDOA/FDOA values depend on:

Emitter Parms: (x_e, y_e, z_e) and transmit frequency f_e $\quad \mathbf{x}_e = [\, x_e \ \ y_e \ \ z_e \ \ f_e \,]^T$

Receivers' Nav Data (positions & velocities): The totality of it called $\mathbf{x}_r$

$$\mathbf{r} = \mathbf{s}(\mathbf{x}_e; \mathbf{x}_r) + \boldsymbol{\varepsilon}$$

- Deterministic "Signal" + Gaussian Noise
- "Signal" is nonlinearly related to parms

To complete the model… we need to know how $s(x_e;x_r)$ depends on x_e and x_r.
Thus we need to find TDOA & FDOA as functions of x_e and x_r

Next

TDOA/FDOA Measurement Model (cont.)

Here we'll simplify to the x-y plane… extension is straight-forward.

Two Receivers with: (x_1, y_1, Vx_1, Vy_1) and (x_2, y_2, Vx_2, Vy_2)
Emitter with: (x_e, y_e)

(Let R_i be the range between Receiver i and the emitter; c is the speed of light.)

The TDOA and FDOA are given by:

$$s_1(x_e, y_e) = \tau_{12} = \frac{R_1 - R_2}{c}$$

$$= \frac{1}{c}\left(\sqrt{(x_1 - x_e)^2 + (y_1 - y_e)^2} - \sqrt{(x_2 - x_e)^2 + (y_2 - y_e)^2}\right)$$

$$s_2(x_e, y_e, f_e) = \nu_{12} = \frac{f_e}{c}\frac{d}{dt}(R_1 - R_2)$$

$$= \frac{f_e}{c}\left[\frac{(x_1 - x_e)Vx_1 + (y_1 - y_e)Vy_1}{\sqrt{(x_1 - x_e)^2 + (y_1 - y_e)^2}} - \frac{(x_2 - x_e)Vx_2 + (y_2 - y_e)Vy_2}{\sqrt{(x_2 - x_e)^2 + (y_2 - y_e)^2}}\right]$$

Next

CRLB for Geo-Location via TDOA/FDOA

Recall: For the General Gaussian Data case the CRLB depends on a FIM that has structure like this:

$$[\mathbf{J}(\boldsymbol{\theta})]_{nm} = \underbrace{\left[\frac{\partial \boldsymbol{\mu}_{\mathbf{x}}(\boldsymbol{\theta})}{\partial \theta_n}\right]^T \mathbf{C}_{\mathbf{x}}^{-1}(\boldsymbol{\theta}) \left[\frac{\partial \boldsymbol{\mu}_{\mathbf{x}}(\boldsymbol{\theta})}{\partial \theta_m}\right]}_{\text{variability of mean w.r.t. parms}} + \underbrace{\frac{1}{2}\mathrm{tr}\left[\mathbf{C}_{\mathbf{x}}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{C}_{\mathbf{x}}(\boldsymbol{\theta})}{\partial \theta_n} \mathbf{C}_{\mathbf{x}}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{C}_{\mathbf{x}}(\boldsymbol{\theta})}{\partial \theta_m}\right]}_{\text{variability of cov. w.r.t. parms}}$$

Here we have a deterministic "signal" plus Gaussian noise so we only have the 1st term… Using the notation introduced here gives…

$$\mathbf{C}_{CRLB}(\mathbf{x}_e) = \left[\underbrace{\frac{\partial \mathbf{s}^T(\mathbf{x}_e)}{\partial \mathbf{x}_e}}_{H^T} \mathbf{C}^{-1} \underbrace{\frac{\partial \mathbf{s}(\mathbf{x}_e)}{\partial \mathbf{x}_e}}_{H} \right]^{-1} \qquad (\bigstar)$$

Called the "Jacobian" … for the 3-D location with TDOA/FDOA will be a $2N \times 4$ matrix whose columns are derivatives of s w.r.t. each of the 4 parameters.

Next

CRLB for Geo-Loc. via TDOA/FDOA (cont.)

TDOA/FDOA Jacobian:

$$\mathbf{H} \triangleq \frac{\partial \mathbf{s}(\mathbf{x}_e)}{\partial \mathbf{x}_e} = \begin{bmatrix} \frac{\partial s_1(\mathbf{x}_e)}{\partial x_e} & \frac{\partial s_1(\mathbf{x}_e)}{\partial y_e} & \frac{\partial s_1(\mathbf{x}_e)}{\partial z_e} & \frac{\partial s_1(\mathbf{x}_e)}{\partial f_e} \\ \frac{\partial s_2(\mathbf{x}_e)}{\partial x_e} & \frac{\partial s_2(\mathbf{x}_e)}{\partial y_e} & \frac{\partial s_2(\mathbf{x}_e)}{\partial z_e} & \frac{\partial s_2(\mathbf{x}_e)}{\partial f_e} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial s_{2N}(\mathbf{x}_e)}{\partial x_e} & \frac{\partial s_{2N}(\mathbf{x}_e)}{\partial y_e} & \frac{\partial s_{2N}(\mathbf{x}_e)}{\partial z_e} & \frac{\partial s_{2N}(\mathbf{x}_e)}{\partial f_e} \end{bmatrix}$$

$$\uparrow \frac{\partial}{\partial x_e} \qquad \uparrow \frac{\partial}{\partial y_e} \qquad \uparrow \frac{\partial}{\partial z_e} \qquad \uparrow \frac{\partial}{\partial f_e}$$

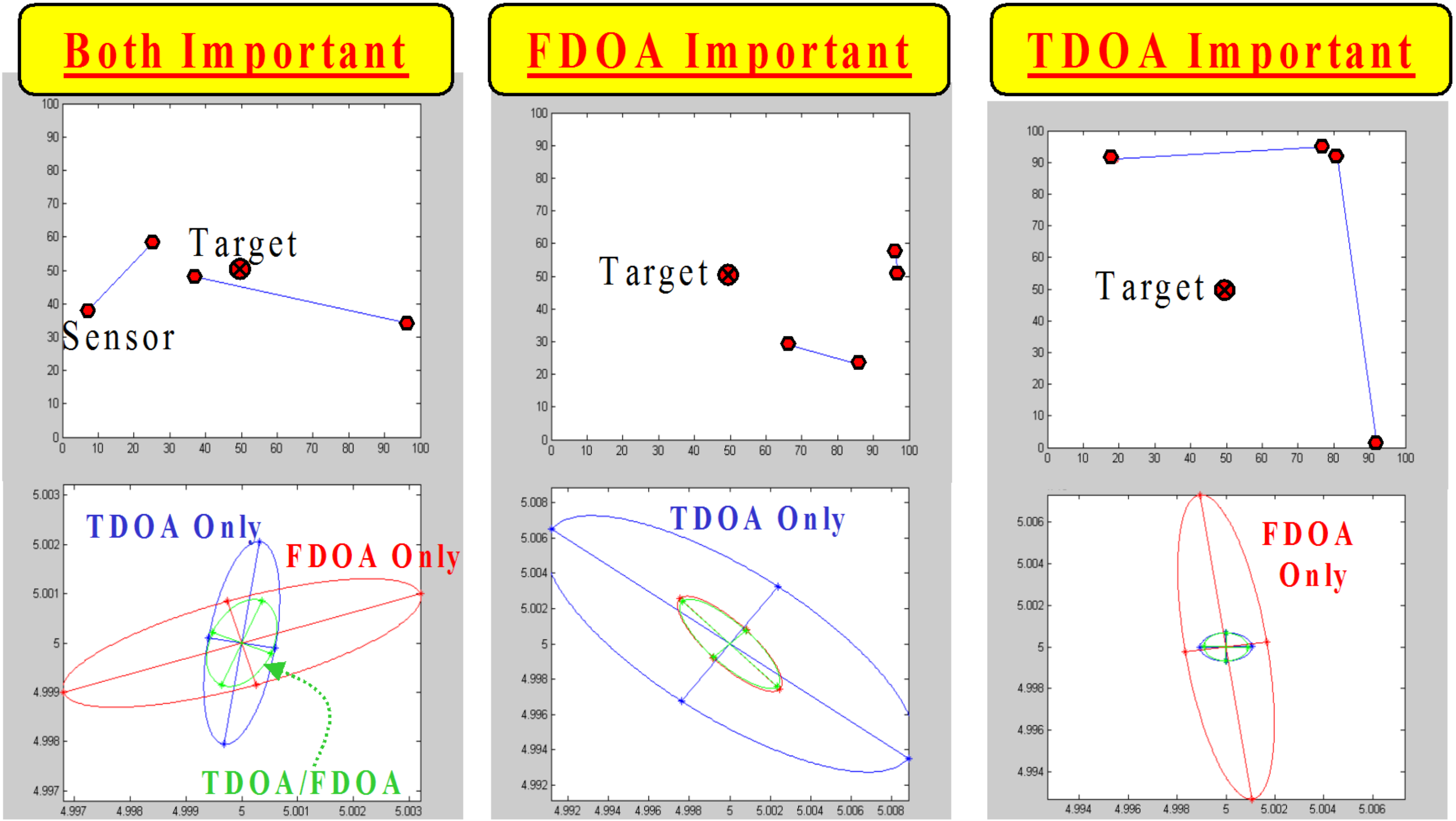
Jacobian can be computed for any desired Rx-Emitter Scenario
Then… plug it into (★) to compute the CRLB for that scenario:

$$\mathbf{C}_{CRLB}(\mathbf{x}_e) = \left[\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H}\right]^{-1}$$

Next

CRLB for Geo-Loc. via TDOA/FDOA (cont.)

Geometry and TDOA vs. FDOA Trade-Offs



Next

Estimator for Geo-Location via TDOA/FDOA

Because we have used the ML estimator to get the TDOA/FDOA estimates the ML's asymptotic properties tell us that we have Gaussian TDOA/FDOA measurements

Because the TDOA/FDOA measurement model is nonlinear it is unlikely that we can find a truly optimal estimate… so we again resort to the ML. For the ML of a Nonlinear Signal in Gaussian we generally have to proceed numerically.

One way to do Numerical MLE is ML Newton-Raphson (need vector version):

$$\hat{\boldsymbol{\theta}}_{k+1} = \hat{\boldsymbol{\theta}}_k - \left\{ \left[\frac{\partial^2 \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} \right]^{-1} \frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right\} \Bigg|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_k}$$

Hessian: $p \times p$ matrix

Gradient: $p \times 1$ vector

However, the "Hessian" requires a second derivative…

This can add complexity in practice… Alternative:

Gauss-Newton Nonlinear Least Squares based on linearizing the model.