<u>**Doppler Tracking**</u>

Passive Tracking of an Airborne Radar:

An Example of Least Squares

“State” Estimation

Problem Statement

Airborne radar to be located follows a trajectory $X(t)$, $Y(t)$, $Z(t)$ with velocities $V_x(t)$, $V_y(t)$, $V_z(t)$

It is transmitting a radar signal whose carrier frequency is f_o.

unknown

Signal is intercepted by a non-moving receiver at known location X_p, Y_p, Z_p.

Problem: Estimate the trajectory $X(t)$, $Y(t)$, $Z(t)$

Solution Here:

- Measure received frequency at instants $t_1, t_2, \dots, t_N$
- Assume a simple model for the aircraft's motion
- Estimate model parameters to give estimates of trajectory

An Admission

This problem is somewhat of a “rigged” application…

- Unlikely it would be done in practice just like this
- Because it will lead to poorly observable parameters
- The **H** matrix is likely to be less than full rank

In real practice we would likely need either:

- Multiple Doppler sensors or
- A single sensor that can measure other things in addition to Doppler (e.g., bearing).

We present it this way to maximize the similarity to the example of locating a non-moving radar from a moving platform

- Can focus on the main characteristic that arises when the parameter to be estimated is a varying function (i.e. state estimation).

Doppler Shift Model

Relative motion between the emitter and receiver… Doppler Shift

Frequency observed at time t is related to the unknown transmitted frequency of f_o by:

$$f(t) = f_o - \frac{f_o}{c}\left[\frac{V_x(t)\big(X_p - X(t)\big) + V_y(t)\big(Y_p - Y(t)\big) + V_z\big(Z_p - Z(t)\big)}{\sqrt{\big(X_p - X(t)\big)^2 + \big(Y_p - Y(t)\big)^2 + \big(Z_p - Z(t)\big)^2}}\right]$$

We measure this at time instants $t_1, t_2, \ldots, t_N$

$$\tilde{f}(t_i) = f(t_i) + v(t_i)$$

Frequency Measurement "Noise"

And group them into a measurement vector:

$$\tilde{\mathbf{f}} = \left[\tilde{f}(t_1)\ \ \tilde{f}(t_2)\ \ \cdots\ \ \tilde{f}(t_N)\right]^T$$

But what are we trying to estimate from this data vector???

Trajectory Model

We can't estimate arbitrary trajectory functions… like $X(t)$, $Y(t)$, etc.

Need a trajectory model…

to reduce the problem to estimating a few parameters

Here we will choose the simplest… **Constant-Velocity Model**

Velocity Values

$$X(t) = V_x \times (t_N - t) + X_N$$

$$Y(t) = V_y \times (t_N - t) + Y_N$$

$$Z(t) = V_z \times (t_N - t) + Z_N$$

Final Positions in Observation Block

Now, given measurements of frequencies $f(t_1), f(t_2), \ldots, f(t_N)$ …

…we wish to **estimate the 7-parameter vector**:

$$\mathbf{x} = [X_N \; Y_N \; Z_N \; V_x \; V_y \; V_z \; f_o]^T$$

Measurement Model and Estimation Problem

Substituting the Trajectory Model into the Doppler Model gives our **measurement model**:

$$f(t,\mathbf{x}) = f_o - \frac{f_o}{c}\left[\frac{V_x\left(X_p - \left[V_x \times (t_N - t) + X_N\right]\right) + V_y(t)\left(Y_p - \left[V_y \times (t_N - t) + Y_N\right]\right) + V_z\left(Z_p - \left[V_z \times (t_N - t) + Z_N\right]\right)}{\sqrt{\left(X_p - \left[V_x \times (t_N - t) + X_N\right]\right)^2 + \left(Y_p - \left[V_y \times (t_N - t) + Y_N\right]\right)^2 + \left(Z_p - \left[V_z \times (t_N - t) + Z_N\right]\right)^2}}\right]$$

Dependence on parameter vector

$$\tilde{f}(t,\mathbf{x}) = f(t,\mathbf{x}) + v(t)$$

Noisy Frequency Measurement

$$\tilde{\mathbf{f}}(\mathbf{x}) = \left[\tilde{f}(t_1,\mathbf{x}) \;\; \tilde{f}(t_2,\mathbf{x}) \;\; \cdots \;\; \tilde{f}(t_N,\mathbf{x})\right]^T$$
$$= \mathbf{f}(\mathbf{x}) + \mathbf{v}$$

Noisy Measurement Vector

Noise-Free Frequency Vector

Noise Vector

Estimation Problem

Given:

Noisy Data Vector: $\tilde{\mathbf{f}}(\mathbf{x}) = \left[\tilde{f}(t_1,\mathbf{x})\ \tilde{f}(t_2,\mathbf{x}) \cdots \tilde{f}(t_N,\mathbf{x})\right]^T$

Sensor Position: X_p, Y_p, Z_p

Estimate:

Parameter Vector: $\mathbf{x} = [X_N\ Y_N\ Z_N\ V_x\ V_y\ V_z\ \ f_o]^T$

"Nuisance" parameter

This is a nonlinear problem…

Although we could use ML to attack this we choose LS here partly because we aren't given an explicit noise model and partly because LS is "easily" applied here!!!

Linearize the Nonlinear Model

We have a non-linear measurement model here…

so we choose to **linearize our model** (as before):

$$\tilde{\mathbf{f}}(\mathbf{x}) \approx \mathbf{f}(\hat{\mathbf{x}}_n) + \mathbf{H}\left[\mathbf{x} - \hat{\mathbf{x}}_n\right] + \mathbf{v}$$

where…

$\hat{\mathbf{x}}_n$ is the "current" estimate of the parameter vector

$\mathbf{f}(\hat{\mathbf{x}}_n)$ is the "predicted" frequency measurements computed using the Doppler & Trajectory models with "back-propagation" (see next)

$$\mathbf{H} = \frac{\partial}{\partial \mathbf{x}} f(\mathbf{t}, \mathbf{x})\Big|_{\mathbf{x}=\hat{\mathbf{x}}_n} = \left[\mathbf{h}_1 \,|\, \mathbf{h}_2 \,|\, \mathbf{h}_3 \,|\, \mathbf{h}_4 \,|\, \mathbf{h}_5 \,|\, \mathbf{h}_6 \,|\, \mathbf{h}_7\right]$$

is the $N\times 7$ Jacobian matrix evaluated at the current estimate

Back-Propagate to Get Predicted Frequencies

Given the current parameter estimate:

$$\hat{\mathbf{x}}_n = [\hat{X}_N(n) \quad \hat{Y}_N(n) \quad \hat{Z}_N(n) \quad \hat{V}_x(n) \quad \hat{V}_y(n) \quad \hat{V}_z(n) \quad \hat{f}_o(n)]^T$$

Back-Propagate to get the current trajectory estimate:

$$\hat{X}_n(t) = \hat{V}_x(n) \times (t_N - t) + \hat{X}_N(n)$$

$$\hat{Y}_n(t) = \hat{V}_y(n) \times (t_N - t) + \hat{Y}_N(n)$$

$$\hat{Z}_n(t) = \hat{V}_z(n) \times (t_N - t) + \hat{Z}_N(n)$$

Use Back-Propagated trajectory to get predicted frequencies:

$$f(t_i, \hat{\mathbf{x}}_n) = \hat{f}_o(n) - \frac{\hat{f}_o(n)}{c} \left[\frac{\hat{V}_x(n)\left(X_p - \hat{X}_n(t)\right) + \hat{V}_y(n)\left(Y_p - \hat{Y}_n(t)\right) + \hat{V}_z(n)\left(Z_p - \hat{Z}_n(t)\right)}{\sqrt{\left(X_p - \hat{X}_n(t)\right)^2 + \left(Y_p - \hat{Y}_n(t)\right)^2 + \left(Z_p - \hat{Z}_n(t)\right)^2}} \right]$$

Converting to Linear LS Problem Form

From the linearized model and the back-propagated trajectory estimate we get:

$$\underbrace{\Delta \mathbf{f}(\hat{\mathbf{x}}_n)}_{\tilde{\mathbf{f}}(\mathbf{x})-\mathbf{f}(\hat{\mathbf{x}}_n)} \approx \mathbf{H}\underbrace{\Delta \mathbf{x}_n}_{\mathbf{x}-\hat{\mathbf{x}}_n} + \mathbf{v}$$

"Residual" Vector

"Update" Vector

This is in standard form of Linear LS… so the solution is:

$$\Delta \hat{\mathbf{x}}_n = \left(\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H}\right)^{-1} \mathbf{H}^T \mathbf{R}^{-1} \Delta \mathbf{f}(\hat{\mathbf{x}}_n)$$

R is the covariance matrix of the measurements

This LS estimated "update" is then used to get an **updated parameter estimate**:

$$\hat{\mathbf{x}}_{n+1} = \hat{\mathbf{x}}_n + \Delta \hat{\mathbf{x}}_n$$

Iterating to the Solution

- n=0: Start with some initial estimate
- Loop until stopping criterion satistfied
  - $n \leftarrow n+1$
  - Compute Back-Propagated Trajectory
  - Compute Residual
  - Compute Jacobian
  - Compute Update
  - Check Update for smallness of norm
    - If Update small enough… stop
    - Otherwise, update estimate and loop