# Pre-Chapter 10
# Results for Two Random Variables

See Reading Notes
posted on BB

Let $X$ and $Y$ be two RVs each with there own PDF: $p_X(x)$ and $p_Y(y)$

Their complete probabilistic description is captured in…

**<u>Joint PDF of *X* and *Y*</u>**: $p_{XY}(x,y)$

Describes probabilities of joint events concerning X and Y.

$$\boxed{\Pr\{(a < X < b) \text{ and } (c < Y < d)\} = \int_a^b \int_c^d p_{XY}(x,y)\,dx\,dy}$$

**<u>Marginal PDFs of *X* and *Y*</u>**: The individual PDFs $p_X(x)$ and $p_Y(y)$

Imagine "adding  up" the joint PDF along one direction of a piece of paper to give values "along one of the <u>margins</u>".

$$\boxed{p_X(x) = \int p_{XY}(x,y)\,dy \qquad p_Y(y) = \int p_{XY}(x,y)\,dx}$$

**Expected Value of Functions of *X* and *Y***: You sometimes create a new RV that is a function of the two of them: $Z = g(X,Y)$.

$$\boxed{E\{Z\} = E_{XY}\{g(X,Y)\} = \int\int g(x,y)\,p_{XY}(x,y)\,dx\,dy}$$

Example: $Z = X + Y$

$$E\{Z\} = E_{XY}\{X+Y\} = \int\int (x+y)\,p_{XY}(x,y)\,dx\,dy$$

$$= \int\int x\,p_{XY}(x,y)\,dx\,dy + \int\int y\,p_{XY}(x,y)\,dx\,dy$$

$$= \int x\left[\int p_{XY}(x,y)\,dy\right]dx + \int y\left[\int p_{XY}(x,y)\,dx\right]dy$$

$$= \int x\,p_X(x)\,dx + \int y\,p_Y(y)\,dy$$

$$= E_X\{X\} + E_Y\{Y\}$$

**Conditional PDFs** : If you know the value of one RV how is the remaining RV now distributed?

$$\boxed{p_{Y|X}(y\,|\,x) = \begin{cases} \dfrac{p_{XY}(x,y)}{p_X(x)}, & p_X(x) \neq 0 \\ 0, & \text{otherwise} \end{cases}}$$

$$\boxed{p_{X|Y}(x\,|\,y) = \begin{cases} \dfrac{p_{XY}(x,y)}{p_Y(y)}, & p_Y(y) \neq 0 \\ 0, & \text{otherwise} \end{cases}}$$

Sometimes we think of a *specific* numerical value upon which we are conditioning… $p_{Y|X}(y|X = 5)$

Other times it is an arbitrary value…

$$\underbrace{p_{Y|X}(y|X = x) \;\text{ or }\; p_{Y|X}(y|x) \;\text{ or }\; p_{Y|X}(y|X)}_{\text{Various Notations}}$$

**Independence**: RVs $X$ and $Y$ are said to be independent if knowledge of the value of one does not change the PDF model for the other.

$$\boxed{\begin{aligned} p_{Y|X}(y \mid x) &= p_Y(y) \\ p_{X|Y}(x \mid y) &= p_X(x) \end{aligned}}$$

This implies (and is implied by)… $p_{XY}(x, y) = p_X(x) p_Y(y)$

$$\boxed{\begin{aligned} p_{Y|X}(y \mid x) &= \frac{p_X(x) p_Y(y)}{p_X(x)} = p_Y(y) \\ p_{X|Y}(x \mid y) &= \frac{p_X(x) p_Y(y)}{p_Y(y)} = p_X(x) \end{aligned}}$$

**Decomposing the Joint PDF**: Sometimes it is useful to be able to write the joint PDF in terms of conditional and marginal PDFs.

From our results for conditioning above we get…

$$\boxed{p_{XY}(x,y) = p_{Y|X}(y \mid x)\, p_X(x)}$$

$$\boxed{p_{XY}(x,y) = p_{X|Y}(x \mid y)\, p_Y(y)}$$

From this we can get results for the marginals:

$$\boxed{\begin{aligned} p_X(x) &= \int p_{X|Y}(x \mid y)\, p_Y(y)\, dy \\ p_Y(y) &= \int p_{Y|X}(y \mid x)\, p_X(x)\, dx \end{aligned}}$$

**Bayes' Rule**: Sometimes it is useful to be able to write one conditional PDF in terms of the other conditional PDF.

$$\boxed{\begin{aligned} p_{Y|X}(y\,|\,x) &= \frac{p_{X|Y}(x\,|\,y)\,p_Y(y)}{p_X(x)} \\[2em] p_{X|Y}(x\,|\,y) &= \frac{p_{Y|X}(y\,|\,x)\,p_X(x)}{p_Y(y)} \end{aligned}}$$

Some alternative versions of Bayes' rule can be obtained by writing the marginal PDFs using some of the above results:

$$\boxed{\begin{aligned} p_{Y|X}(y\,|\,x) &= \frac{p_{X|Y}(x\,|\,y)\,p_Y(y)}{\int p_{XY}(x,y)dy} = \frac{p_{X|Y}(x\,|\,y)\,p_Y(y)}{\int p_{X|Y}(x\,|\,y)\,p_Y(y)dy} \\[2em] p_{X|Y}(x\,|\,y) &= \frac{p_{Y|X}(y\,|\,x)\,p_X(x)}{\int p_{XY}(x,y)dx} = \frac{p_{Y|X}(y\,|\,x)\,p_X(x)}{\int p_{Y|X}(y\,|\,x)\,p_X(x)dx} \end{aligned}}$$

**<u>Conditional Expectations</u>**: Once you have a conditional PDF it works EXACTLY like a PDF… that is because it ***<u>IS</u>*** a PDF!

Remember that any expectation involves a function of a random variable(s) times a PDF and then integrating that product.

So the trick to working with expected values is to make sure you know three things:

1. What function of which RVs
2. What PDF
3. What variable to integrate over

# For conditional expectations… one idea but several notations!

$$E_{X|Y}\{g(X,Y)\} = \int g(x,y)\,p_{X|Y}(x\,|\,y)\,dx$$

Uses subscript on *E* to indicate that you use the cond. PDF.

Does *<u>not</u>* explicitly state the <u>value</u> at which *Y* should be fixed so use an arbitrary *y*

$$E_{X|Y=y_o}\{g(X,Y)\} = \int g(x,y_o)\,p_{X|Y}(x\,|\,y_o)\,dx$$

Uses subscript on *E* to indicate that you use the cond. PDF.

Explicitly states that the <u>value</u> at which *Y* should be fixed is $y_o$

$$E\{g(X,Y)\,|\,Y\} = \int g(x,y)\,p_{X|Y}(x\,|\,y)\,dx$$

Uses "conditional bar" inside brackets of *E* to indicate use of the cond. PDF.

Does *<u>not</u>* explicitly state the <u>value</u> at which *Y* should be fixed so use an arbitrary *y*

$$E\{g(X,Y)\,|\,Y=y_o\} = \int g(x,y_o)\,p_{X|Y}(x\,|\,y_o)\,dx$$

Uses "conditional bar" inside brackets of *E* to indicate use of the cond. PDF.

Explicitly states that the <u>value</u> at which *Y* should be fixed is $y_o$

**<u>Decomposing Joint Expectations</u>**: When averaging over the joint PDF it is sometimes useful to be able to decompose it into nested averaging in terms of conditional and marginal PDFs.

This uses the results for decomposing joint PDFs.

$$E\{g(X,Y)\} = E_{XY}\{g(X,Y)\}$$

$$E\{g(X,Y)\} = \int\int g(x,y) \underbrace{p_{XY}(x,y)}_{p_{Y|X}(y|x)\,p_X(x)} dxdy$$

$$= \int_x \underbrace{\left[\int_y g(x,y)p_{Y|X}(y\,|\,x)\right]}_{E_{Y|X}\{g(X,Y)\}} p_X(x)dx$$

$$= E_X\left\{\underbrace{E_{Y|X}\{g(X,Y)\}}\right\}$$

This is an RV that "inherits" the PDF of X!!!

# Ex. Decomposing Joint Expectations:

$$E\{g(X,Y)\} = E_X\left\{E_{Y|X}\{g(X,Y)\}\right\}$$

Let $X$ = # on Red Die  $Y$ = # on Blue Die  $g(X,Y) = X + Y$

| $X/Y$ | 1 | 2 | 3 | 4 | 5 | 6 | $E\{Y\|X\}$ |
|---|---|---|---|---|---|---|---|
| 1 | (1+1) | (1+2) | (1+3) | (1+4) | (1+5) | (1+6) | $\sum_{y=1}^{6}(1+y)\frac{1}{6}=4.5$ |
| 2 | (2+1) | (2+2) | (2+3) | (2+4) | (2+5) | (2+6) | $\sum_{y=1}^{6}(2+y)\frac{1}{6}=5.5$ |
| 3 | (3+1) | (3+2) | (3+3) | (3+4) | (3+5) | (3+6) | $\sum_{y=1}^{6}(3+y)\frac{1}{6}=6.5$ |
| 4 | (4+1) | (4+2) | (4+3) | (4+4) | (4+5) | (4+6) | $\sum_{y=1}^{6}(4+y)\frac{1}{6}=7.5$ |
| 5 | (5+1) | (5+2) | (5+3) | (5+4) | (5+5) | (5+6) | $\sum_{y=1}^{6}(5+y)\frac{1}{6}=8.5$ |
| 6 | (6+1) | (6+2) | (6+3) | (6+4) | (6+5) | (6+6) | $\sum_{y=1}^{6}(6+y)\frac{1}{6}=9.5$ |

**These constitute an RV with uniform probability of 1/6**

$$E\{X+Y\} = E\{E\{Y \mid X\}\} = \sum_{x=1}^{6} E\{Y \mid x\}\frac{1}{6} = 7$$