Wiener Filter for Deterministic Blur Model

Based on Ch. 5 of Gonzalez & Woods, *Digital Image Processing, 2nd Ed.*, Addison-Wesley, 2002

One common application of the Wiener filter has been in the area of simultaneous de-blurring and de-noising of an image.

From Section 5.5:

$f(x, y)$ is the original image (2-D signal of x, y spatial variables)

Observed image is $$g(x,y) = H\left[f(x,y)\right] + n(x,y)$$

Observed Image · Blur Operator · Original Image · 2-D Random Noise

If there were only blurring…

- seek to find inverse of H

If there were only noise…

- seek a filter that passes image & removes some noise

The Wiener filter seeks to optimally balance these two issues!

LSI Blur Model (Section 5.5)

Also called Linear Position-Invariant (LPI)

Common model for the blur operator is Linear Shift-Invariant (LSI):

$$H\left[a_1 f_1(x,y) + a_2 f_2(x,y)\right] = a_1 H\left[f_1(x,y)\right] + a_2 H\left[f_2(x,y)\right]$$ **Linearity**

$$H\left[f_{in}(x,y)\right] = f_{out}(x,y) \Rightarrow H\left[f_{in}(x-\alpha, y-\beta)\right] = f_{out}(x-\alpha, y-\beta)$$ **Shift Inv.**

An LSI system can be described by its impulse response:

$$H\left[\delta(x,y)\right] \triangleq h(x,y)$$

Then the output is expressed as a 2-D convolution:

$$H\left[f(x,y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\alpha, y-\beta)\, f(\alpha,\beta)\, d\alpha\, d\beta$$

This is a continuous-space version … it is possible to do the same for discrete-space version

Now our blurred image model is:

$$f_{blur}(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(x-\alpha,y-\beta)\,f(\alpha,\beta)\,d\alpha\,d\beta$$

Taking the 2-D Fourier transform of the above model gives

$$F_{blur}(u,v)=H(u,v)F(u,v)$$

Where the 2-D Fourier transforms are given by

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$H(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(x,y)e^{-j2\pi(ux+vy)}dxdy$$

Frequency Response of Blur

Note that in principle if we know $H(u,v)$ then we can use the **inverse filter** to recover the original image

$$F(u,v)=\frac{F_{blur}(u,v)}{H(u,v)}$$

Blur Caused by Planar Motion (see Section 5.6.3)

A common blur model is that due to planar motion while the camera's aperture is open during time interval $[0,T]$.

This means that during $[0,T]$ the image moves in the x and y directions according to functions $x_0(t)$ and $y_0(t)$

Then our observed image model is:

$$g(x,y) = \int_{-\infty}^{\infty} f\left(x - x_0(t), y - y_0(t)\right) dt + n(x,y)$$

Converting to frequency domain gives (see Gonzalez & Woods for steps):

$$G(u,v) = H(u,v)F(u,v) + N(u,v)$$

with

$$H(u,v) = \int_0^T e^{-j2\pi\left[ux_0(t)+vy_0(t)\right]} dt$$

Blur Caused by Uniform Linear Motion (see Section 5.6.3)

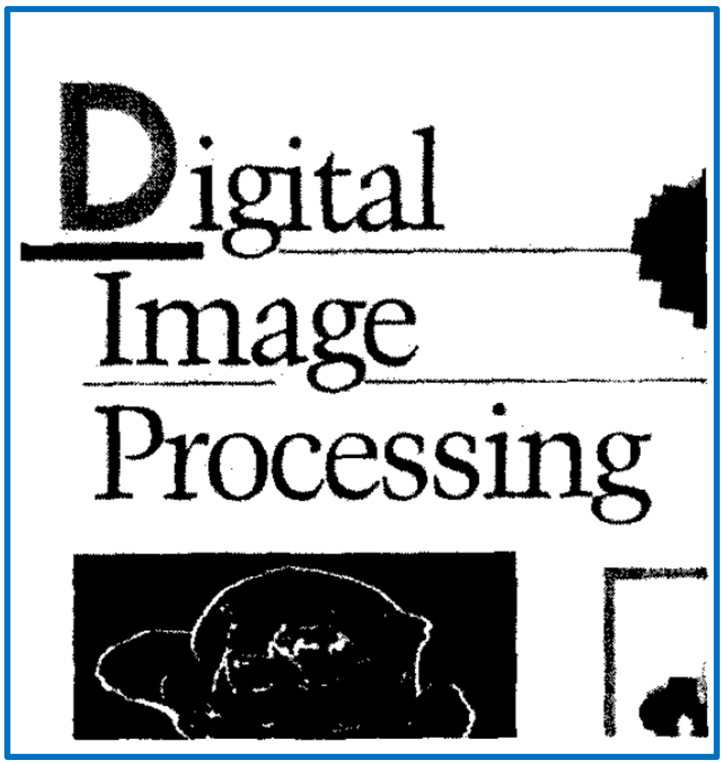
This is planar motion with constant speed in each direction:

$$x_0(t) = at/T \qquad \text{and} \qquad y_0(t) = bt/T$$

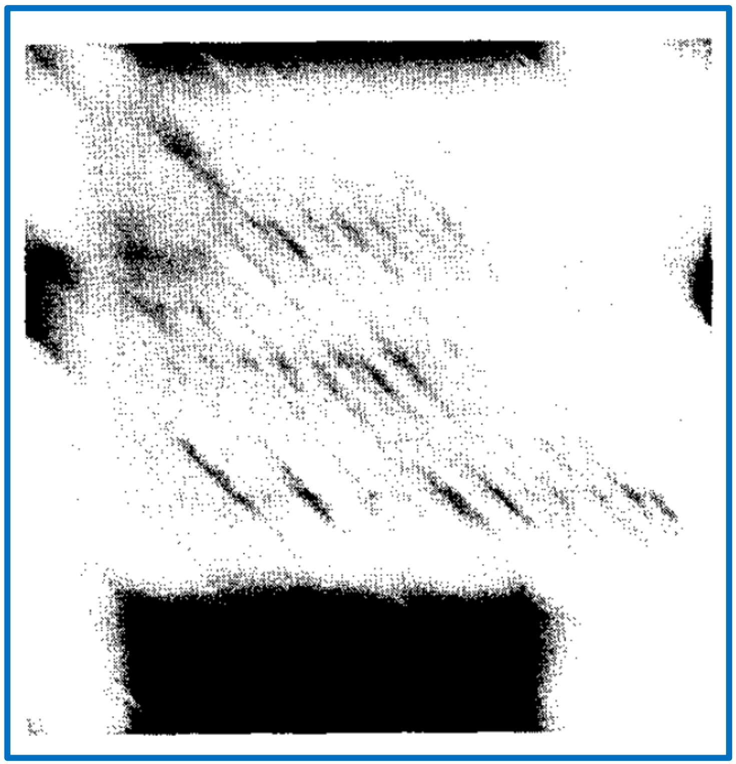
Then the blur's frequency response is two sincs:

$$H(u,v) = \left[T \frac{\sin\left[\pi(ua+vb)\right]}{\pi(ua+vb)} e^{-j\pi(ua+vb)} \right]$$

So this is like a 2-D lowpass filter!



Original Image



Blurred Image w/ $a = b = 0.1, T = 1$

Blurred and Noisy Image

Now our observed image model is:

2-D WSS Noise Process

$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\alpha,y-\beta)\,f(\alpha,\beta)\,d\alpha\,d\beta+n(x,y)$$

Taking the 2-D Fourier transform of the above model gives

$$G(u,v)=H(u,v)F(u,v)+N(u,v)$$

Solving this using the "Inverse Filtering" Viewpoint gives

$$\hat{F}(u,v)=\frac{G(u,v)}{H(u,v)}=F(u,v)+\frac{N(u,v)}{H(u,v)}$$

This can exacerbate the noise when $H(u,v)$ is 0 or small at some frequencies!

The inverse filter focuses on the de-blurring…
The Wiener filter seeks to optimally balance these two issues.

We will solve the Wiener filter in this Frequency-Domain view

Wiener Filter for Blurred & Noisy Images (see Sect. 5.8)

Without going into the details… we will borrow the frequency-domain view we saw for the IIR Wiener smoother. Recall….

$$x[n] = s[n] + w[n]$$

No Blurring

$$\hat{S}(f) = \left[\frac{P_{ss}(f)}{P_{ss}(f) + P_{ww}(f)} \right] X(f)$$

Accounting for the blurring and the change to 2-D gives

$$G(u,v) = H(u,v)F(u,v) + N(u,v)$$

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)P_{ff}(u,v)}{|H(u,v)|^2 P_{ff}(u,v) + P_{nn}(u,v)} \right] G(u,v)$$

Must Know!

- $P_{ff}(u,v)$ = **Power Spectral Density of Image**
- $P_{nn}(u,v)$ = **Power Spectral Density of Noise**

Bayesian!! Thing we are estimating is modeled as random!

In practice our knowledge needed varies:

- $H(u,v)$ might be reasonably well known
- $P_{nn}(u,v)$ might be quite well known
- $P_{ff}(u,v)$ might NOT be known at all!

E.g., white noise
$P_{nn}(u,v) = N_o$
(constant)

Then re-arranging gives
$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{\left|H(u,v)\right|^2 + N_o / P_{ff}(u,v)}\right] G(u,v)$$

A common trick is to replace this term by a constant *K*:

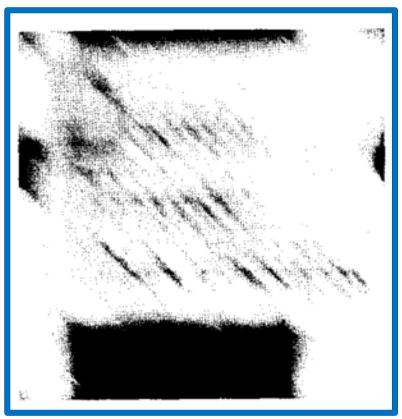
$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{\left|H(u,v)\right|^2 + K}\right] G(u,v)$$

then the value of *K* is chosen interactively to give the best result. (So this is an "off-line" approach).

With enough attention to detail, this approach can be implemented using the FFT algorithm and applied to discrete images.

Results for Wiener Filter (sect. 5.8 of Gonzalez and Woods)

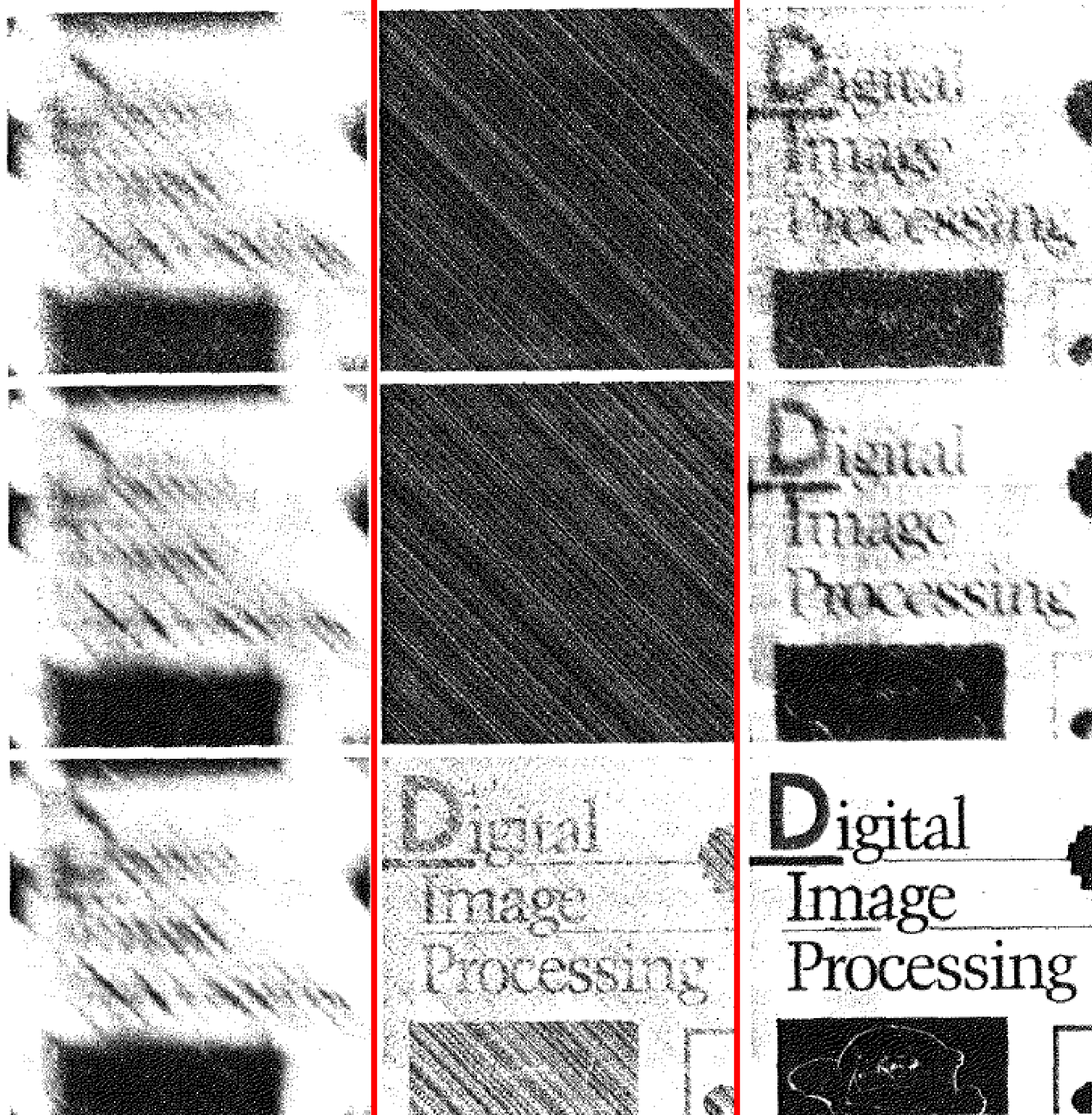
Original Image



Uniform Blurred Image
w/ $a = b = 0.1, T = 1$

Increasing Noise Variance

Blurred & Noisy | **Inverse Filter** | **Wiener Filter**



Iterative Wiener Filter (Not in Gonzalez and Woods)

Start by using the above solution:

$$\hat{F}_0(u,v) = \left[\frac{H^*(u,v)}{\left|H(u,v)\right|^2 + K} \right] G(u,v)$$

Now use that estimate image to estimate the PSD of the image

$$\hat{F}_0(u,v) \Rightarrow P_{\hat{f}\hat{f},0}(u,v)$$

Lots of literature on how to estimate the PSD of a WSS process

Re-Estimate image using

$$\hat{F}_1(u,v) = \left[\frac{H^*(u,v)}{\left|H(u,v)\right|^2 + N_o / P_{ff,0}(u,v)} \right] G(u,v)$$

Iterate:

$$\hat{F}_n(u,v) \Rightarrow P_{\hat{f}\hat{f},n}(u,v)$$

$$\hat{F}_{n+1}(u,v) = \left[\frac{H^*(u,v)}{\left|H(u,v)\right|^2 + N_o / P_{ff,n}(u,v)} \right] G(u,v)$$

Alternate View (not in Gonzalez and Woods)

An alternate view is a discrete-image, space-domain approach:

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \mathbf{n}$$

g & **n** vectors defined similarly

H is blurring matrix formed to provide equivalent of 2D conv

f is vector formed by stacking columns of discrete image

R_{ff} = Correlation Matrix of Image's Vector f

R_{nn} = Correlation Matrix of Noise's Vector n

Recall our general Wiener smoothing with vector parameter:

$$\mathbf{x} = \mathbf{s} + \mathbf{w} \quad \Rightarrow \quad \underset{[N\times N]}{\hat{\mathbf{s}} = \mathbf{R}_{ss}}\underset{[N\times N]}{(\mathbf{R}_{ss} + \mathbf{R}_{ww})^{-1}}\underset{[N\times 1]}{\mathbf{x}}$$

Modifying this to handle the blurring matrix gives

$$\hat{\mathbf{f}} = \mathbf{R}_{\mathbf{ff}}\mathbf{H}^T\left[\mathbf{H}\mathbf{R}_{\mathbf{ff}}\mathbf{H}^T + \mathbf{R}_{\mathbf{nn}}\right]^{-1}\mathbf{g}$$

Compare to previous Freq. Domain result:

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)P_{ff}(u,v)}{|H(u,v)|^2 P_{ff}(u,v) + P_{nn}(u,v)}\right]G(u,v)$$

Iterative Version of Alternate View (not in Gonzalez and Woods)

Initial Filtering:

- Use g to get estimate $\hat{\mathbf{R}}_{\mathbf{gg}}$
- Use it to approximate $\hat{\mathbf{R}}_{\mathbf{ff},\mathbf{0}} = \hat{\mathbf{R}}_{\mathbf{gg}}$
- Use that in place of $\mathbf{R}_{\mathbf{ff}}$ in filter to get $\hat{\mathbf{f}}_{\mathbf{0}}$

Iteration for n = 1, 2, 3, …:

- Use $\hat{\mathbf{f}}_{\boldsymbol{n}-\mathbf{1}}$ to estimate $\hat{\mathbf{R}}_{\mathbf{ff},\boldsymbol{n}+\mathbf{1}}$
- Use $\hat{\mathbf{R}}_{\mathbf{ff},\boldsymbol{n}+\mathbf{1}}$ in filter to get $\hat{\mathbf{f}}_{\boldsymbol{n}}$