Ch. 2 Math Preliminaries for Lossless Compression

Section 2.4 Coding

Some General Considerations

Definition: An *Instantaneous Code* maps each symbol into a codeword

Notation: $a_i \to \phi(a_i)$

Ex. 1:

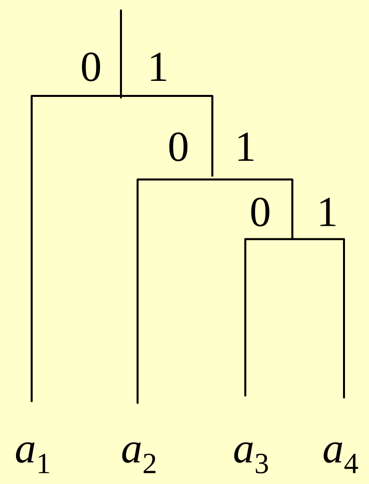
$$a_1 \to 0$$
$$a_2 \to 1$$
$$a_3 \to 00$$
$$a_4 \to 11$$

For Ex. 1: $\phi(a_3) = 00$

Ex. 2:

$$a_1 \to 0$$
$$a_2 \to 10$$
$$a_3 \to 110$$
$$a_4 \to 111$$

This code has a tree structure:



What characteristics must a code ϕ have?

Unambiguous (UA): For $a_i \neq a_j$, $\phi(a_i) \neq \phi(a_j)$

The codes in Ex. 1 and Ex.2 each are UA

Is UA enough?? No! Consider Ex. 1 coding two different source sequences:

$a_1 \; a_2 \; a_1 \; a_1 \; a_2 \; a_2$

$a_1 \; a_2 \; a_3 \; a_4$

$a_1 \; a_2 \; a_1 \; a_1 \; a_2 \; a_2$

They each get coded to the bit stream: **0 1 0 0 1 1**

$a_1 \; a_2 \; a_3 \; a_4$

Can't uniquely decode this bit sequence!!

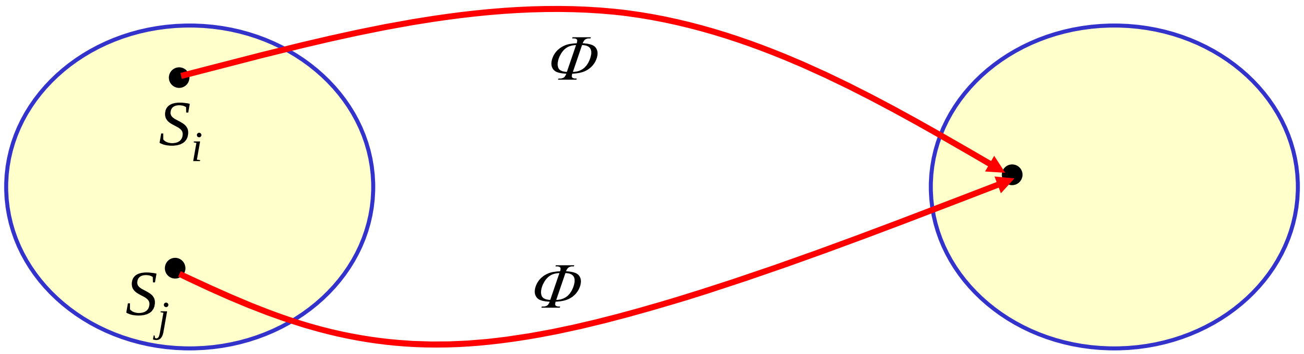
So… UA guarantees that can decode each symbol by itself but not necessarily a stream of coded symbols!!

Define mapping of sequences under code ϕ

$$\Phi(\underbrace{a_{i_1}\ a_{i_2}\ a_{i_3}\ a_{i_4}\ \ldots a_{i_N}}_{S_i}) = \underbrace{\phi(a_{i_1})\ \phi(a_{i_2})\ \phi(a_{i_3})\ \phi(a_{i_4}) \ldots \phi(a_{i_N})}_{\text{Concatenation of code words}}$$

Concatenation of code words

Don't want two sequences of symbols to map to the same bit stream:



Source Sequence Space

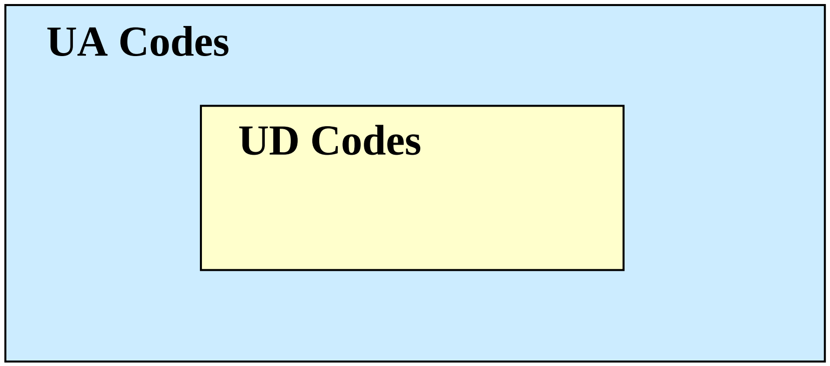
Code Sequence Space

Leads to need for…

Uniquely Decodable (UD): Let S_i & S_j be two sequences from the same source (not necessarily of the same length).

Then code ϕ is UD if the only way that $\Phi(S_i) = \Phi(S_j)$ is for $S_i \neq S_j$

Does UD → UA??? **YES!**



Then UD is enough??? **YES!**

But in practice it is helpful to restrict to a subset of UD codes called "Prefix Codes".

Prefix Code: A UD code in which no codeword may be the prefix of another codeword.

Ex. 2 above is a prefix code:

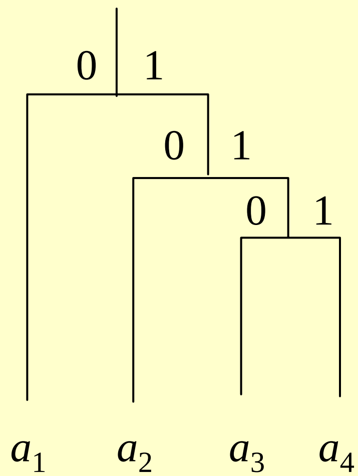
Ex. 2:

$a_1 \rightarrow 0$

$a_2 \rightarrow 10$

$a_3 \rightarrow 110$

$a_4 \rightarrow 111$



This code is not prefix

Ex. 3:

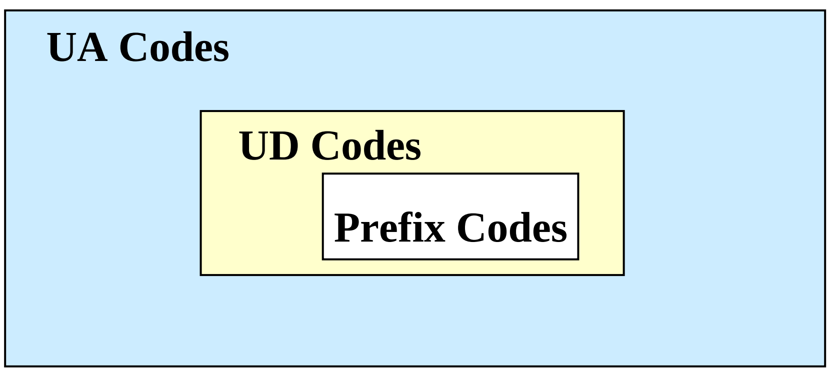
$a_1 \rightarrow 0$

$a_2 \rightarrow 01$

$a_3 \rightarrow 011$

$a_4 \rightarrow 0111$

Does Prefix ➔ UD??? YES!



Do we lose anything by restricting to prefix codes?

No… as we'll see later!

How do we compare various UD codes???

(i.e., What is our measure of performance?)

<u>Average Code Length</u>: Info theory says to use average code length per symbol… For a source with symbols a_1, a_2, … a_N and a code ϕ the average code length is define by

$$\bar{l}(\phi) = \sum_{i=1}^{N} \underbrace{P(a_i)}_{\text{Prob. of } a_i} \underbrace{n\{\phi(a_i)\}}_{\text{\# of bits in codeword for } a_i}$$

Prob. of a_i

# of bits in codeword for a_i

<u>Optimum Code</u>: The UD code with the smallest average code length

<u>Example</u>: For $P(a_1) = ½$ $P(a_2) = ¼$ $P(a_3) = P(a_3) = 1/8$

This source has a entropy of 1.75 bits

Here are three possible codes and their average lengths:

<u>Symbol</u>	**UA Non-UD Code** (Ex. 1)	**Prefix Code** (Ex. 2)	**UD Non-Prefix** (Ex. 3)	**Info of symbol** $-\log_2[P(a_i)]$
a_1	0	0	0	1
a_2	1	10	01	2
a_3	00	110	011	3
a_4	11	111	0111	3
Avg. Length:	**1.25 bits**	**1.75 bits**	**1.875 bits**	**$H(S)$ = 1.75 bits**

Length better than $H(S)$!!

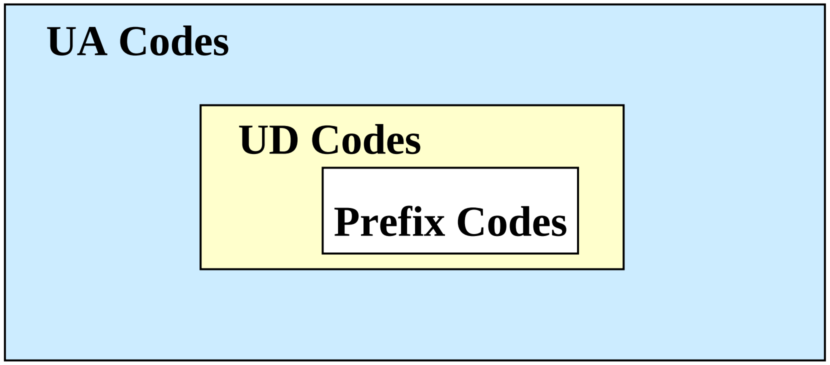
Length equals $H(S)$

Length larger than $H(S)$

<u>BUT</u>… not usable because it is <u>Non</u>-UD

Prefix Code gives smallest usable code!!

Info Theory Says: Optimum Code is <u>always</u> a prefix code!!



The proof of this uses the Kraft-McMillan Inequality which we'll discuss next.

How do we find the optimum prefix code?

(Note: not just any prefix code will be optimum!)

We'll discuss this later….

2.4.3 Kraft-McMillan Inequality

This result tells us that an optimal code can always be chosen to be a prefix code!!! The Theorem has 2 parts….

Theorem Part #1: Let C be a code having N codewords… with codeword lengths of $l_1, l_2, l_3, \ldots, l_N$

If C is uniquely decodable, then $\sum_{i=1}^{N} 2^{-l_i} \leq 1$

For notation: $K(C) \triangleq \sum_{i=1}^{N} 2^{-l_i}$

Proof: Here is the main idea used in the proof…

If $K(C) > 1$, then $[K(C)]^n$ grows exponentially w.r.t. n

So… if we can show that $[K(C)]^n$ grows, say, no more than linearly we have our proof. Thus we need to show that

$$\left[K(C)\right]^n \leq \alpha n + \beta$$

Some constants

For arbitrary integer n:

$$[K(C)]^n = \left[\sum_{i=1}^{N} 2^{-l_i}\right]^n = \left[\sum_{i_1=1}^{N} 2^{-l_{i_1}}\right]\left[\sum_{i_2=1}^{N} 2^{-l_{i_2}}\right]\cdots\left[\sum_{i_n=1}^{N} 2^{-l_{i_n}}\right]$$

Note use of n <u>different</u> dummy variables!

$$= \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_n=1}^{N} 2^{-\left(l_{i_1}+l_{i_2}+\cdots+l_{i_n}\right)} \qquad (\bigstar)$$

Note that this exponent is nothing more than the length of a sequence of selected codewords of code C… Let this be $L(i_1, i_2, i_3, \ldots, i_n)$ and we can re-write ($\bigstar$) as

$$[K(C)]^n = \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_n=1}^{N} 2^{-L(i_1,i_2,\cdots,i_n)} = 2^{-L(1,1,\cdots,1)} + 2^{-L(1,1,\cdots,2)} + \cdots + 2^{-L(N,N,\cdots,N)}$$

The smallest $L(i_1, i_2, i_3, \ldots, i_n)$ can be is n (when each codeword in the sequence is 1 bit long)

The longest $L(i_1, i_2, i_3, \ldots, i_n)$ can be is nl where l is the longest codeword in C.

So then:
$$[K(C)]^n = 2^{-L(1,1,\cdots,1)} + 2^{-L(1,1,\cdots,2)} + \cdots + 2^{-L(N,N,\cdots,N)}$$
$$= A_n 2^{-n} + A_{n+1} 2^{-(n+1)} + \cdots + A_{nl} 2^{-(nl)}$$

$$(\bigstar\bigstar) \qquad [K(C)]^n = \sum_{k=n}^{nl} A_k 2^{-k}$$

A_k = # times $L(i_1, i_2, i_3, \ldots, i_n) = n$

Remember that we are trying to establish this bound: $\left[K(C)\right]^n \le \alpha n + \beta$

we don't need the A_k values exactly… just need a good *upper bound* on them!

First: The # of k-bit binary sequences = 2^k

The "If" part of the theorem!

Second: If our code is uniquely decodable, then each of these can represent one and only one sequence of codewords whose total length = k bits

There may be some in the 2^k that are not valid

$$\Rightarrow \quad A_k \le 2^k$$

We can now use this bound in (★★) to get a bound on $[K(C)]^n$:

$$\left[K(C)\right]^n = \sum_{k=n}^{nl} A_k 2^{-k} \le \sum_{k=n}^{nl} \underbrace{2^k 2^{-k}}_{=1} = nl - n + 1$$

Thus… $[K(C)]^n$ grows slower than exponentially

Hence… $K(C) \le 1$ <End of Proof>

<u>Part #1 says</u>: If code with lengths $\{l_1, l_2, \ldots l_N\}$ is uniquely decodable, then the lengths satisfy the inequality

<u>Part #2 says</u>: Given lengths $\{l_1, l_2, \ldots l_N\}$ that satisfy the inequality, then we can always find a prefix code w/ these lengths

<u>Theorem Part #2</u>: Given integers $\{l_1, l_2, \ldots l_N\}$ such that $\sum_{i=1}^{N} 2^{-l_i} \le 1$

We can always find a <u>prefix</u> code with lengths $\{l_1, l_2, \ldots l_N\}$

<u>Proof</u>: This is a "Proof by Construction": we will show how to construct the desired prefix code. "WLOG"…. Assume that $l_1 \le l_2 \le \ldots \le l_N$

Define the numbers $w_1, w_2, \ldots, w_N$ using

$$w_1 = 0$$

$$w_j = \sum_{i=1}^{j-1} 2^{l_j - l_i}, \quad j > 1$$

Think of this in terms of a
binary representation (see next slide for an example)

Example of Creating the w_j

$$l_1 = 1 \quad l_2 = 3 \quad l_3 = 3 \quad l_4 = 5 \quad l_5 = 5 \qquad \sum_{i=1}^{5} 2^{-l_i} = 0.8125 < 1$$

$$w_1 = 0$$

$$w_2 = \sum_{i=1}^{1} 2^{l_2 - l_i} = 2^{3-1} = 4 = 100_2$$

$$w_3 = \sum_{i=1}^{2} 2^{l_3 - l_i} = 2^{3-1} + 2^{3-3} = 5 = 101_2$$

$$w_4 = \sum_{i=1}^{3} 2^{l_4 - l_i} = 2^{5-1} + 2^{5-3} + 2^{5-3} = 24 = 11000_2$$

$$w_5 = \sum_{i=1}^{4} 2^{l_5 - l_i} = 2^{5-1} + 2^{5-3} + 2^{5-3} + 2^{5-5} = 25 = 11001_2$$

For $j > 1$, the binary representation of w_j uses $\lceil \log_2 w_j \rceil$ bits

Easy to show (see textbook) that: "# bits in w_j" $\le l_j$ ➔ $\lceil \log_2 w_j \rceil \le l_j$, for $j \ge 1$

This is where we use that $\sum_{i=1}^{N} 2^{-l_i} \le 1$

Now use the binary reps of the w_j to construct the prefix codewords having lengths $\{l_1, l_2, \dots, l_N\}$

If

- $\lceil \log_2 w_j \rceil = l_j$ then set j^{th} codeword = binary w_j
- $\lceil \log_2 w_j \rceil < l_j$ then set j^{th} codeword = [binary w_j $\underbrace{0 \dots 0}$]

Append enough 0's to get l_j total bits

So now we've constructed a code with the desired lengths… Is it a prefix code???

Show it is by using contradiction… Assume that it is NOT a prefix code and show that it leads to something that contradicts a known condition…

Suppose that the constructed code is <u>not</u> prefix… thus, for some $j < k$ the codeword C_j is a prefix of codeword C_k…

⇒ (l_j MSBs of w_k) = w_j

⇒ Right-Shift & Chop $w_k = w_j$

⇒ $$\left\lfloor \frac{w_k}{2^{l_k - l_j}} \right\rfloor = w_j \qquad (\bigstar)$$

There is always a "But" in a proof by contradiction!

But… by "design": $$w_k = \sum_{i=1}^{k-1} 2^{l_k - l_i}$$

So see if (★) contradicts this required condition:

Put this w_k into (★) and show that something goes wrong

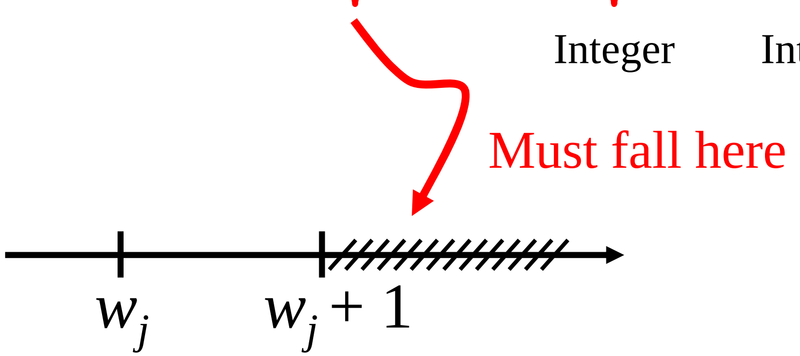
$$(\star) \Rightarrow \quad \frac{w_k}{2^{l_k - l_j}} = \sum_{i=1}^{k-1} 2^{l_j - l_i}$$

$$= \underbrace{\sum_{i=1}^{j-1} 2^{l_j - l_i}} + \sum_{i=j}^{k-1} 2^{l_j - l_i}$$

w_j by Defn

$$= w_j + 2^0 + \sum_{i=j+1}^{k-1} 2^{l_j - l_i} \quad \geq w_j + 1$$

Thus, $\underbrace{\frac{w_k}{2^{l_k - l_j}}} \geq \underbrace{w_j + 1}_{\text{Integer}} > \underbrace{w_j}_{\text{Integer}}$



$$\Rightarrow \left\lfloor \frac{w_k}{2^{l_k - l_j}} \right\rfloor \geq w_j + 1$$

…which contradicts $(\star)$

So code *is* prefix!

<End of Proof>

Meaning of Kraft-McMillan Theorem

Question: So what do these two parts of the theorem tell us???

Answer:

Shortest Avg. Length

- We are looking for the optimal UD code.
- Once we find it we know its codeword lengths satisfy the K-M inequality
  - Part #1 of the theorem tells us that!!!
- Once we have such lengths (that satisfy the K-M ineq.) we can *construct* a prefix code having those optimal lengths…
  - This is guaranteed by Part #2 of the theorem
  - This gives us a prefix code that is optimal!!!

So… everytime we find the optimal code, if it isn't already prefix we can replace it with a prefix code that is just as optimal!

Can focus on finding optimal prefix codes… w/o worrying that we could find a better code that is not prefix!