Ch. 14 Subband Coding

Introduction &
Multirate Background

Introduction

Given signal $x[n]$ to compress…

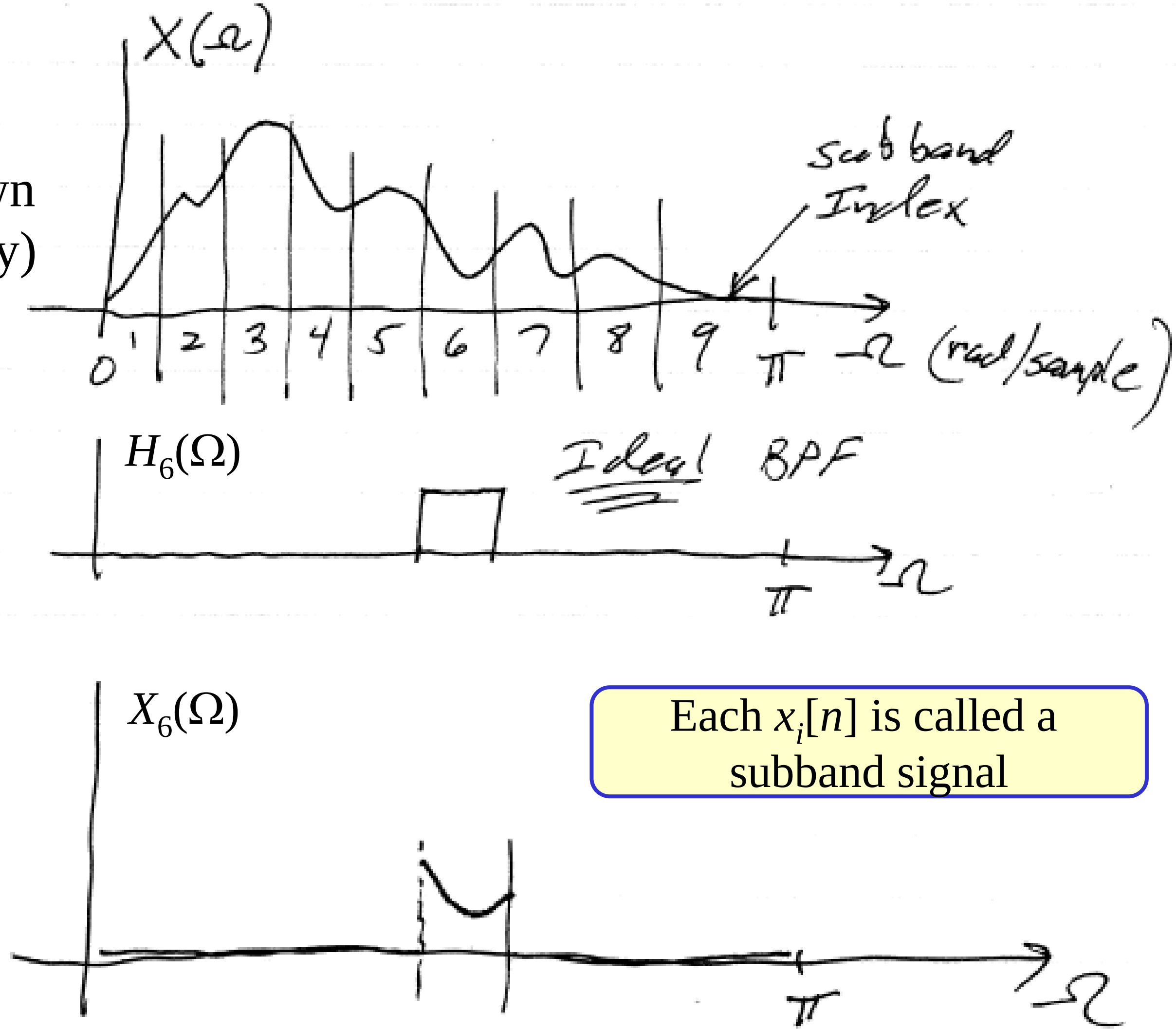
Idea: Split signal into M signals $x_1[n], x_2[n], \ldots, x_M[n]$
such that each signal can be more easily/effectively compressed.

Goal: signals $x_1[n], x_2[n], \ldots, x_M[n]$ should be made such that

- Each $x_i[n]$ is uncorrelated…
  - then using SQ on each is a viable (though still suboptimal) approach
- Some $x_i[n]$ have smaller dynamic range
  - Then can use fewer bits for a given desired distortion
- Should be a clear way to exploit psychological effects (for audio and video) or other effects that imply some $x_i[n]$ are more important

Illustration of Subbands

[-π,0) is not shown
(Recall Symmetry)



Each $x_i[n]$ is called a subband signal

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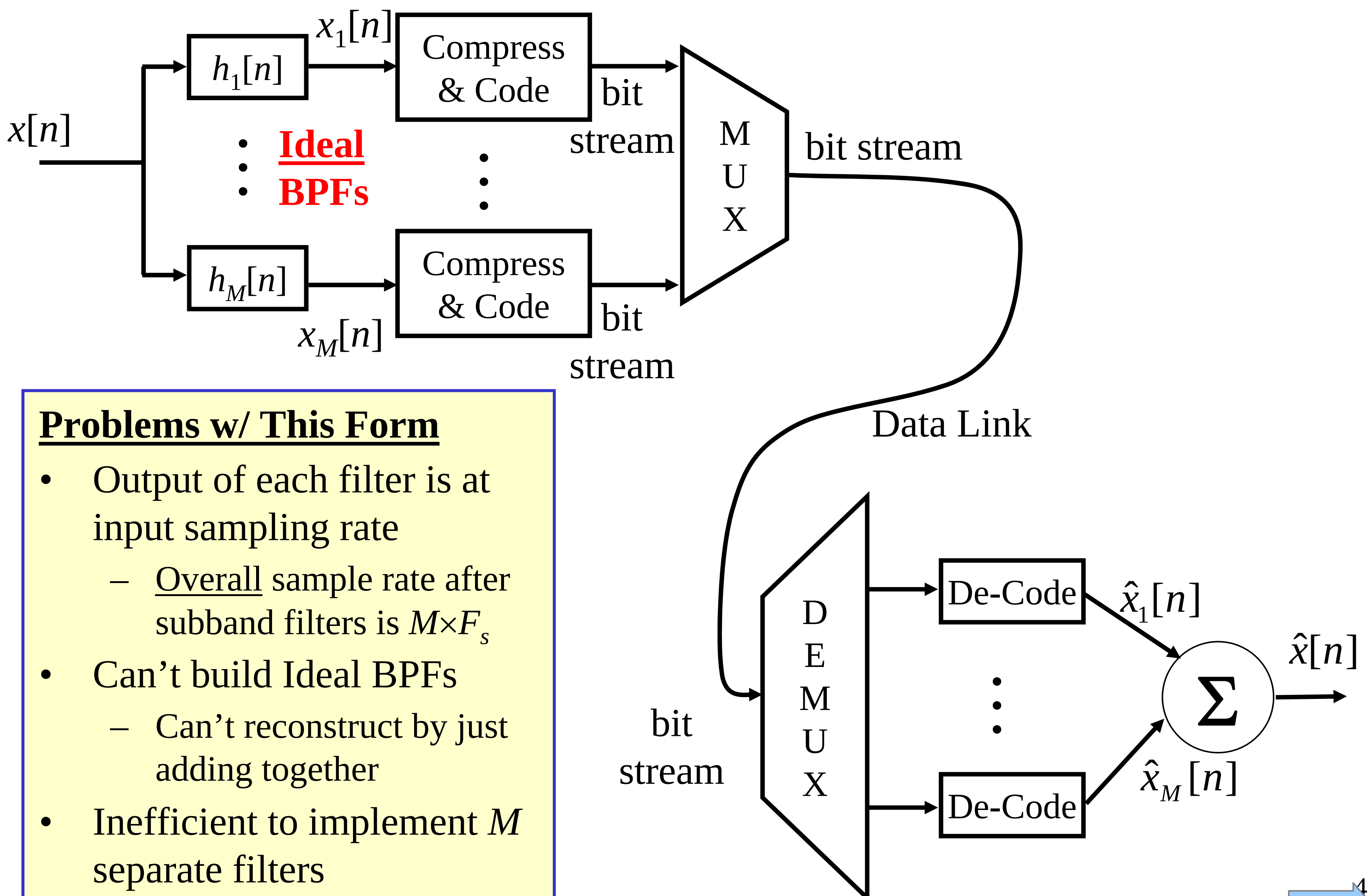
Motivational Form (Not Practical)



Problems w/ This Form

- Output of each filter is at input sampling rate
  - Overall sample rate after subband filters is $M \times F_s$
- Can't build Ideal BPFs
  - Can't reconstruct by just adding together
- Inefficient to implement M separate filters

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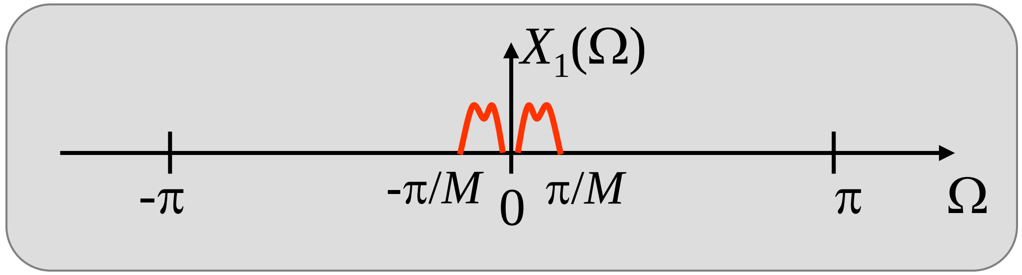
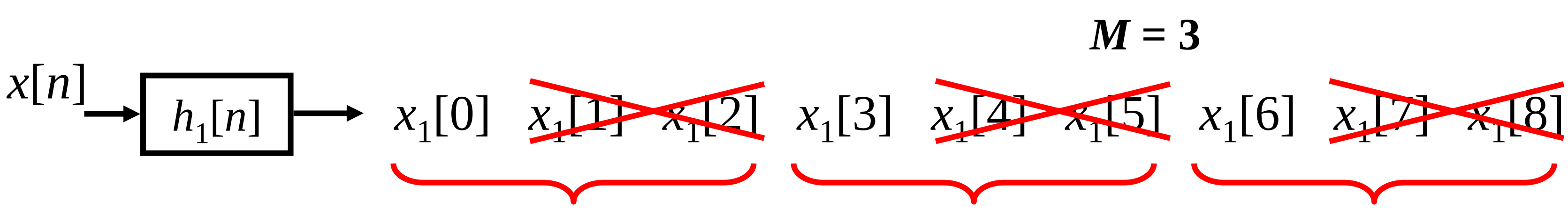
Fixing Sample Rate Problem: Multirate

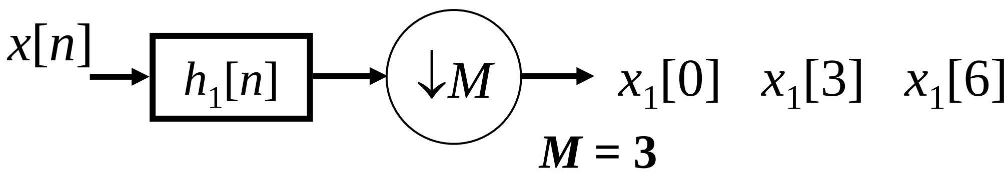
Take a look at $x_1[n]$



This signal is oversampled by a factor of M
(If it were not oversampled it would fill the entire -π to π)

To sample it slower by a factor of M… just throw away M-1 samples out of every M samples (called "**Decimation by *M***")…





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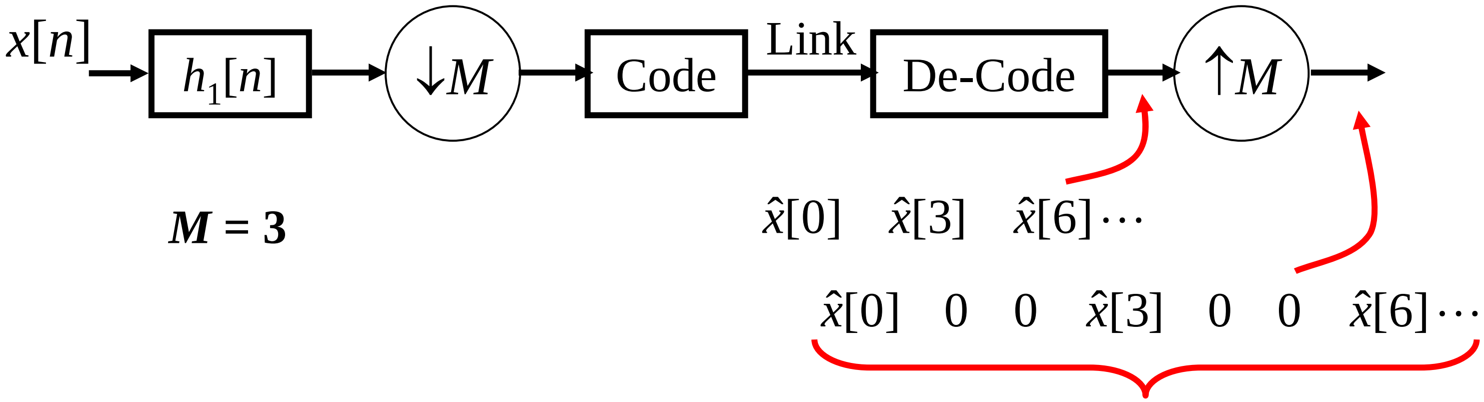
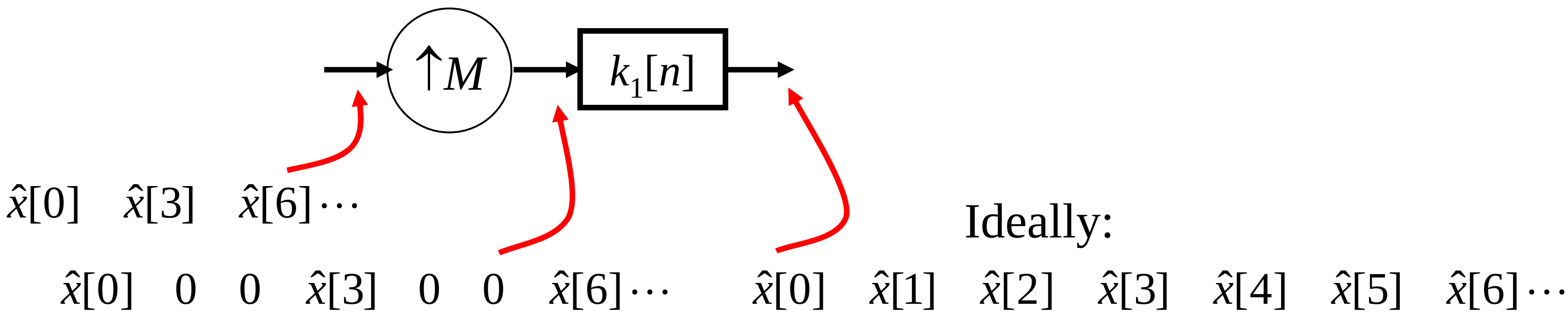
Now… we need someway at the decoder side to get back up to the original sampling rate (called “**Expansion by *M***”)…



Has correct # samples/sec…
But can’t have those zeros!!!

A filter can “smooth out” the jumps due to the zeros (called “**Interpolation**”)….



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Subband Coding System

We can do a similar thing for all the other channels… and the result is:

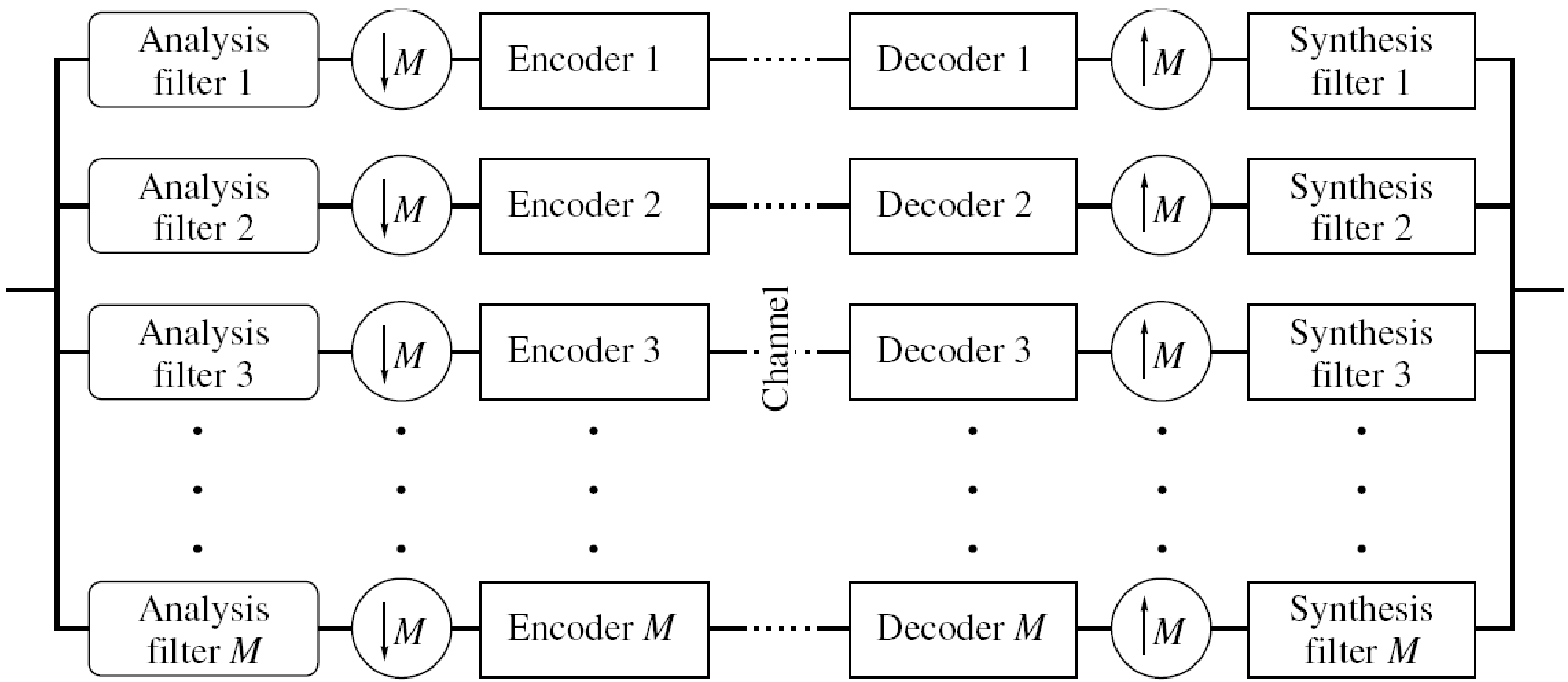

FIGURE 14.7 Block diagram of the subband coding system.

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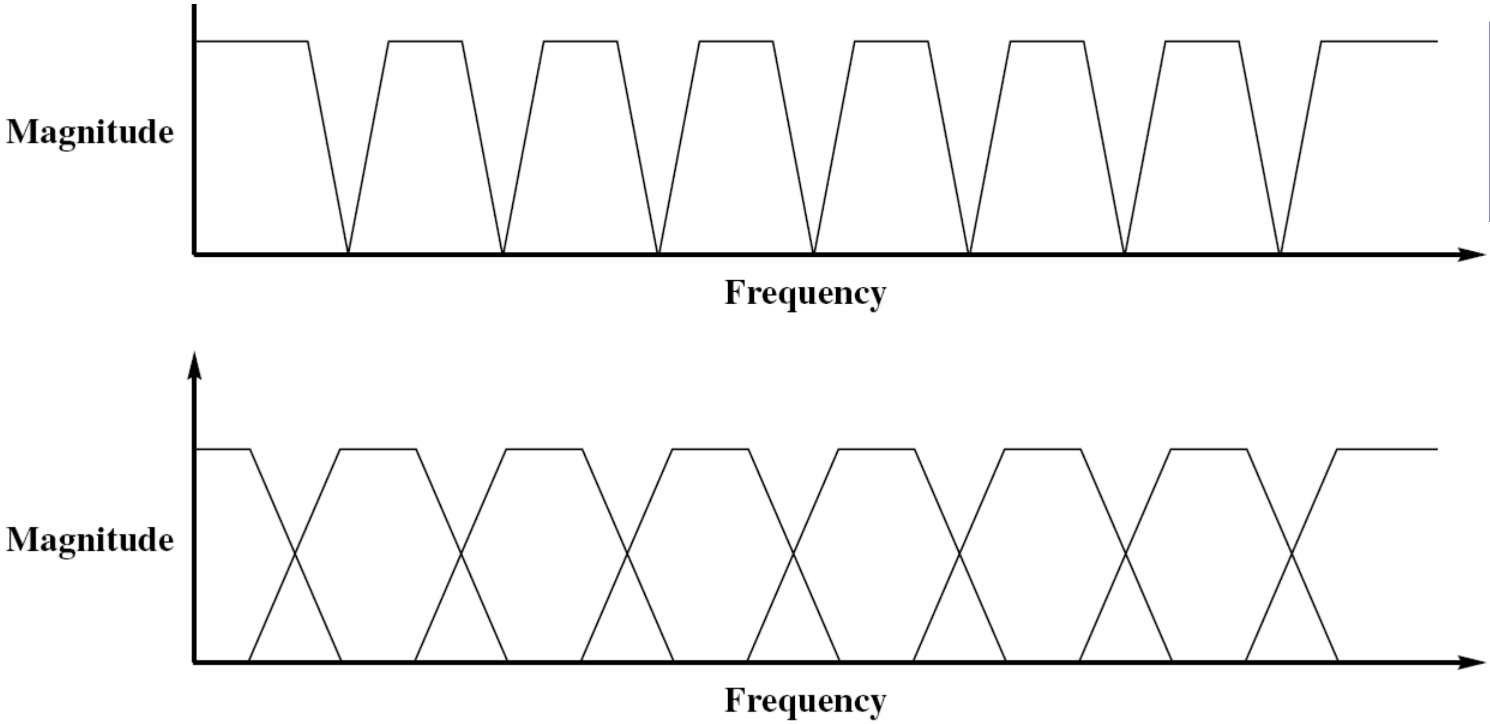
Roughly…What should the analysis filters look like?



Gaps will result in loss of info

FIGURE 14.8 **Nonoverlapping and overlapping filter banks.**

Note: These are not true *achievable* shapes of filter frequency responses

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Subband Coding System Details

Filter Design Goal: If we remove encode/decode… then we want our filters to be designed so that output = input… this is called "***Perfect Reconstruction***".

Analysis Filters must also provide frequency decomposition into essentially non-overlapping subbands… should give "easy to code" signals

Synthesis Filters are chosen to give the desired perfect reconstruction. Their design will depend on the design of the analysis filters.

Encoding/Decoding Goals:

1. Choose methods matched to resulting channel characteristics
2. Allocate bit budget across the channels

Multi-Rate Goal: properly decrease and then restore the sampling rate

Decimation reduces each channel's sample rate to keep the total analysis filter bank's output sample rate equal to the input sample rate

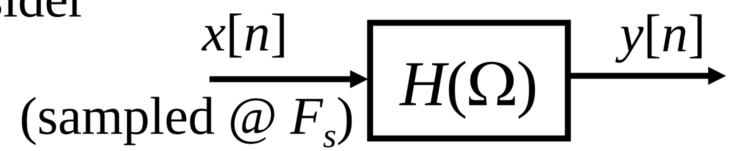
Interpolation returns each channel to original rate before reconstruction

To understand how filter banks work we need to understand:

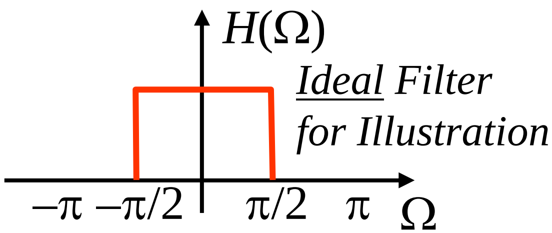
- Effect of decimation on signal spectrum
- Effect of expansion on signal spectrum
- How to choose Analysis & Synthesis filters to achieve perfect reconstruction (PR)

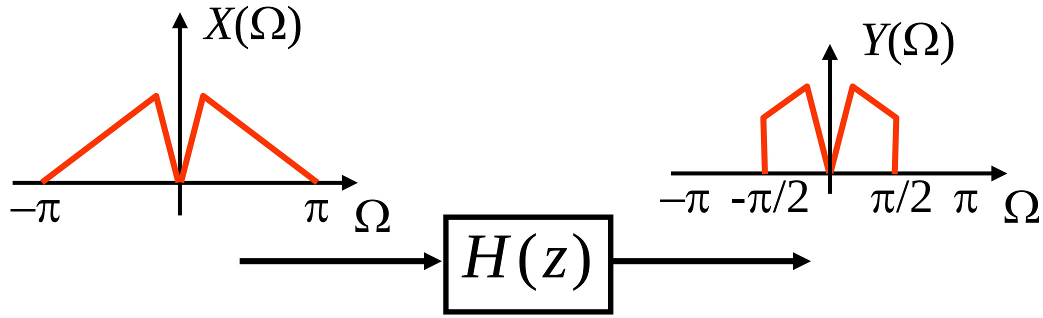
Decimation (Down Sampling)

Consider

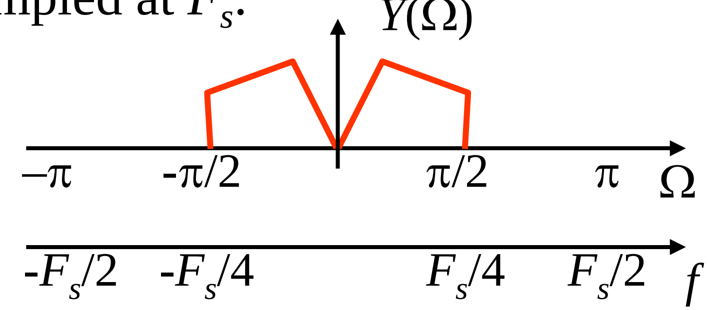


First let $H(\Omega)$ be an ideal LPF w/ cut-off frequency $\Omega = \pi/2$… "half-band filter"

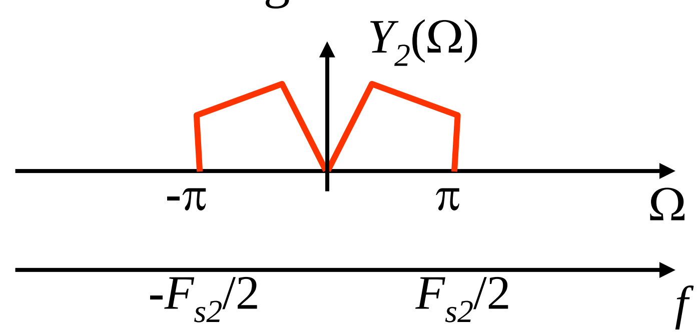




Now… imagine what continuous-time signal would give $y[n]$ if it were sampled at F_s:



So… if we sampled this CT signal at half the original rate (@ $F_{s2}=F_s/2$) then we would get this:



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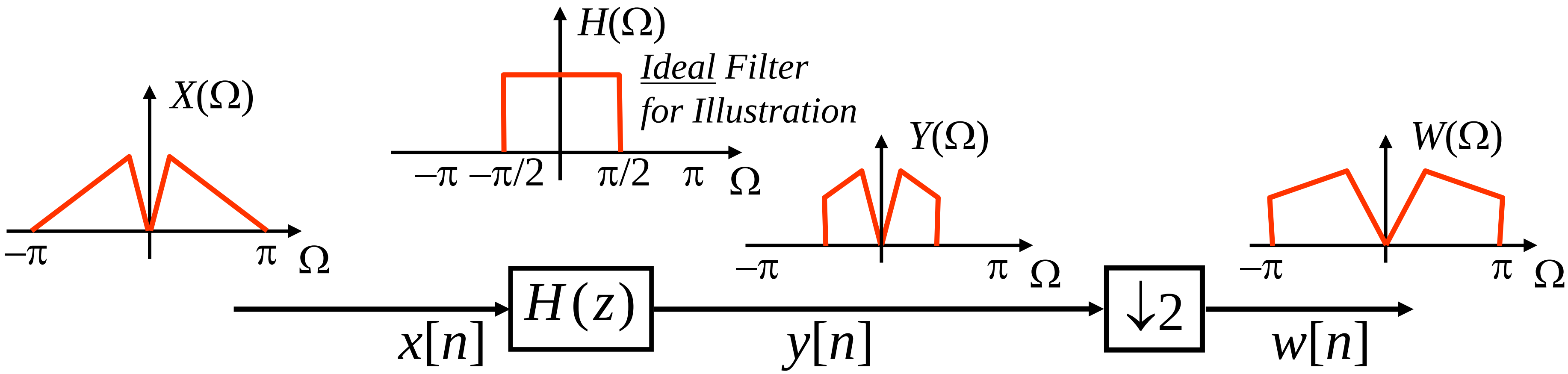
So… just looking at things in the DT domain with the sampling rate change happening because of down-sampling… we should see the SAME thing:



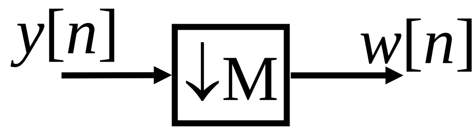
In general… if the filter passes only in the range $\Omega \in [-\pi/M, \pi/M]$ we can downsample by M

Now… all of this was based on "intuition" and for an ***Ideal*** LPF… We need to do a detailed analysis….

Next

Math Analysis of Decimation

Decimation – Time-Domain Math View



***M*-Fold Decimation**

$$w[n] = y[Mn] \quad \text{for} \quad n \in Z$$

For *M*=3

$$w[0] = x[0]$$

$$w[1] = x[3]$$

$$w[2] = x[6]$$

$$\vdots$$

To really understand what is happening we need to look in the frequency domain…

M-Fold Decimation – Frequency-Domain

Notation: $\{Zx_{(\downarrow M)}\}(z) = X^{Z}_{(\downarrow M)}(z) = \{X^{Z}(z)\}_{(\downarrow M)}$

- Similar for DTFT
- Similar for Expansion

Q: What is $X_{(\downarrow M)}(z)$ in terms of $X(z)$???

What do we expect????!!!!

Lower F_s causes Spectral Replicas to Move to Lower Frequencies

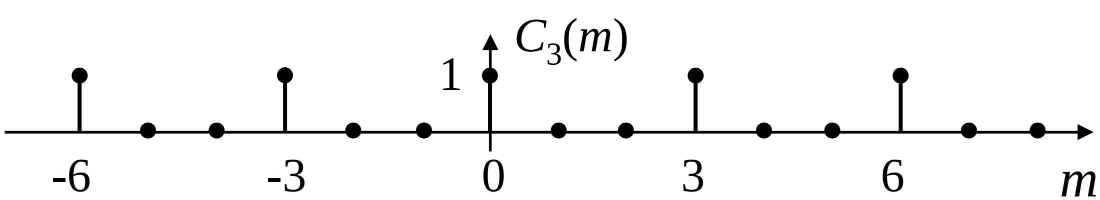
Should look <u>exactly like</u> sampling at a lower F_s

Thus… <u>increased aliasing is possible</u>!!!

To answer this we need to define a useful function ("comb" function):

$$c_M[n] = \sum_{k=-\infty}^{\infty} \delta[n - kM] = \delta[n \bmod M] = \frac{1}{M}\sum_{m=0}^{M-1} W_M^{mn}$$

$$W_M \triangleq e^{j2\pi/M}$$



Call this (★)…
This is like a DT Fourier Series and is easily verified!

Next

M-Fold Decimation – Frequency-Domain (cont.)

Now… use the comb function to write decimation:

$$
\begin{aligned}
x_{(\downarrow M)}[n] &= x[nM] \\
&= x[nM]c_M[nM]
\end{aligned}
$$

Doesn’t Really Do Anything Here… But Later it Will!!

Now… take Z-Transform, using this form:

$$X^{z}_{(\downarrow M)}(z) = \underbrace{\sum_{n=-\infty}^{\infty} x[nM]c_M[nM]z^{-n}}_{\cdots + x[0]z^{0} + x[M]z^{-1} + x[2M]z^{2} + \cdots} = \underbrace{\sum_{n=-\infty}^{\infty} x[n]c_M[n]z^{-n/M}}_{\cdots + x[0]z^{0} + 0 + \cdots + 0 + x[M]z^{-1} + 0 + \cdots}$$

Action of $C_M[n]$

Now… take Z-Transform, using this form:

$$X^{z}_{(\downarrow M)}(z) = \sum_{n=-\infty}^{\infty} x[n]\left[\frac{1}{M}\sum_{m=0}^{M-1} W_M^{mn}\right] z^{-n/M}$$

M-Fold Decimation – Frequency-Domain (cont.)

Now… just manipulate:

$$X^{z}_{(\downarrow M)}(z) = \sum_{n=-\infty}^{\infty} x[n]\left[\frac{1}{M}\sum_{m=0}^{M-1} W_M^{mn}\right] z^{-n/M}$$

$$= \frac{1}{M}\sum_{m=0}^{M-1}\left[\sum_{n=-\infty}^{\infty} x[n]\left(W_M^{-m} z^{1/M}\right)^{-n}\right]$$

$$= \frac{1}{M}\sum_{m=0}^{M-1} X^{z}(W_M^{-m} z^{1/M})$$

ZT of Decimated Signal is…

$$X^{z}_{(\downarrow M)}(z) = \frac{1}{M}\sum_{m=0}^{M-1} X^{z}(W_M^{-m} z^{1/M})$$

M-Fold Decimation – Frequency-Domain (cont.)

Now to see a little better what this says… convert ZT to DTFT.

Recall: DTFT is the ZT evaluated on the unit circle:

$$z = e^{j\theta} \quad \Rightarrow \quad z^{1/M} = e^{j\theta/M}$$

Also, by definition: $W_M^{-m} = e^{-j2\pi m/M}$

Then we get…. DTFT of Decimated Signal is…

$$X^{\mathrm{f}}_{(\downarrow M)}(\theta) = \frac{1}{M}\sum_{m=0}^{M-1} X^{\mathrm{f}}\left(\frac{\theta - 2\pi m}{M}\right)$$

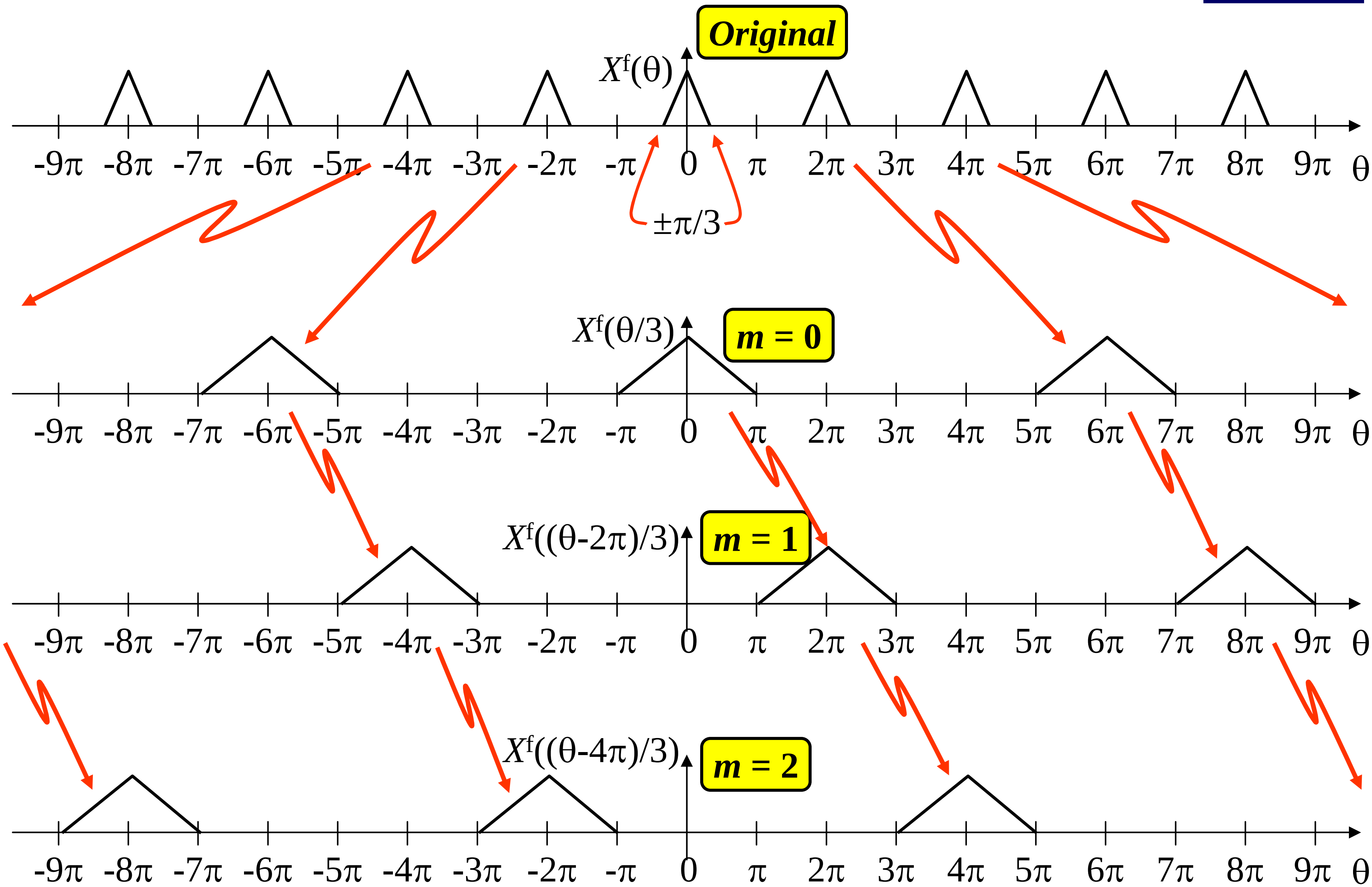
Stretches Spectrum

1. Axis-Scale $X^{\mathrm{f}}(\theta)$ to get $X^{\mathrm{f}}(\theta/M)$ – a stretch
2. Then shift by $2\pi m$ to get new replicas

➔ Decimation Adds Spectral Replicas of Scaled DTFT

Example: DTFT for *M*-Fold Decimation

$M = 3$



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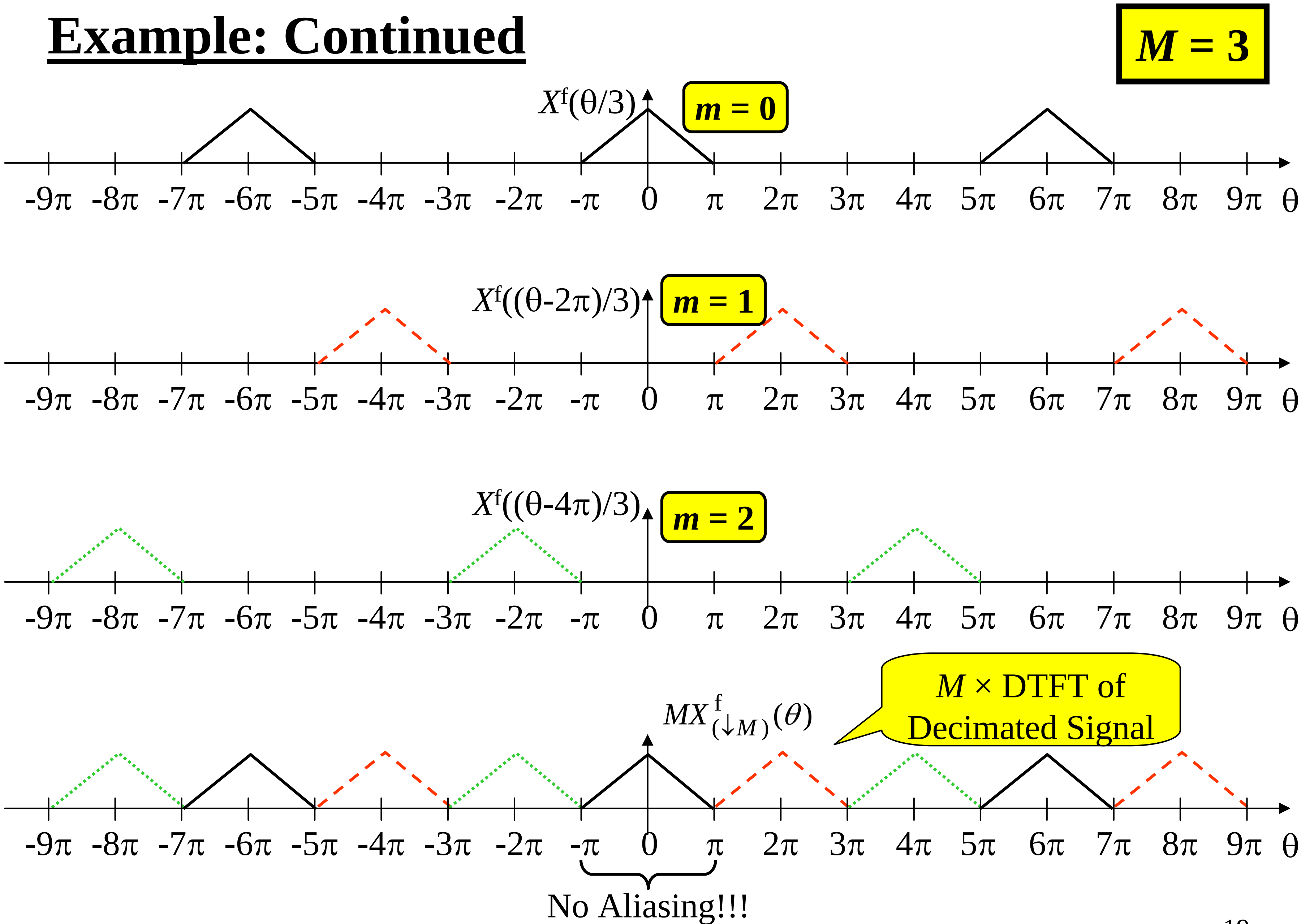
Example: Continued



Example: Insights

1. The *M*-decimated signal will have no aliasing… only if the signal being decimated has: $X^{f}(\theta) = 0 \quad \text{for } |\theta| > \pi / M$

This makes complete sense from an "ordinary" sampling theorem view point!!!

Such a signal is called an "Mth-Band Signal"

2. After *M*-decimating an Mth-band signal, the spectrum of the decimated signal will fill the [-π, π] band.

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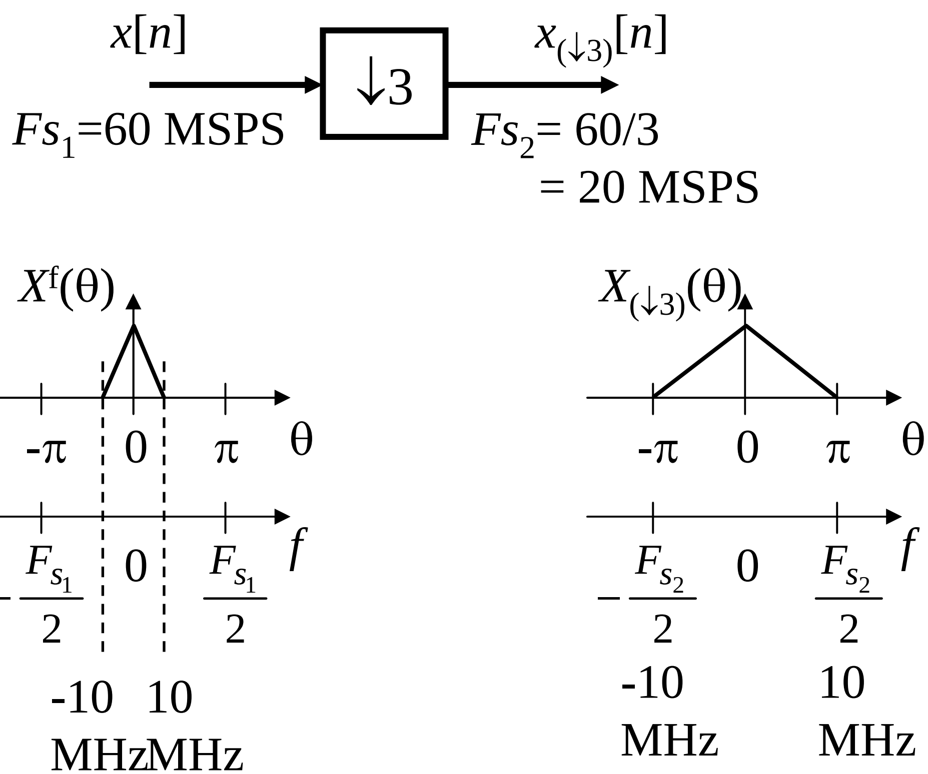
Effect on "Physical" Frequency

Although decimation changes the digital frequency of the signal, the corresponding "physical" frequency is not changed… as the following example shows:



<u>Note</u>
Expansion Also Has No Effect on Physical Frequency

Signal Still Occupies <u>Same</u> Physical Frequency

L-Fold Expansion – Frequency-Domain

Q: What is $X_{(\uparrow L)}(z)$ in terms of $X(z)$???

What do we expect????!!!!

Certainly **<u>NOT</u>** the same as *<u>really</u>* sampling at a higher rate because of the inserted zeros!!!

Frequency Domain analysis answers this!!!

$$X^{z}_{(\uparrow L)}(z) = \sum_{n=-\infty}^{\infty} x_{(\uparrow L)}[n] z^{-n}$$

$$= + \cdots + x[0]z^{0} + \underbrace{0 + \cdots + 0}_{L-1 \text{ zeros}} + x[1]z^{-L} + \underbrace{0 + \cdots + 0}_{L-1 \text{ zeros}} + x[2]z^{-2L}$$

$$= \sum_{n=-\infty}^{\infty} x[n] z^{-Ln} = X^{z}(z^{L})$$

ZT of Expanded Signal is…

$$X^{z}_{(\uparrow L)}(z) = X^{z}(z^{L})$$

Next

L-Fold Expansion – Frequency-Domain (cont.)

Now to see a little better what this says… convert ZT to DTFT.
Recall: DTFT is the ZT evaluated on the unit circle:

$$z = e^{j\theta} \quad \Rightarrow \quad z^L = e^{jL\theta}$$

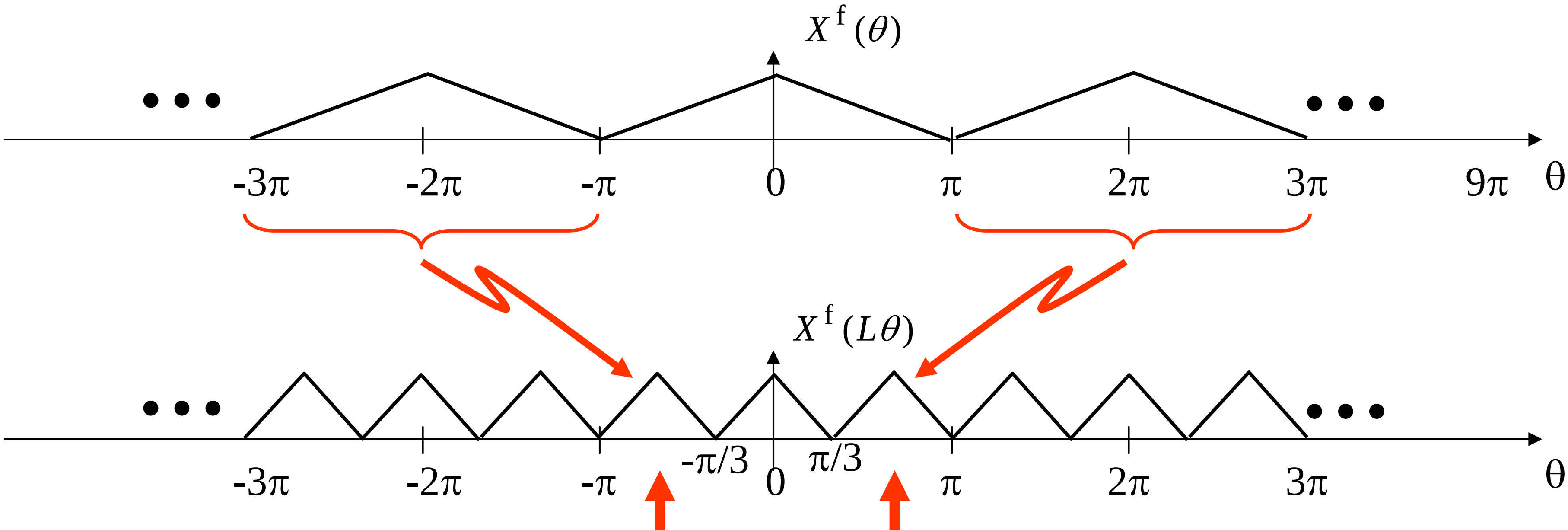
DTFT of Decimated Signal is…

$$X^{\text{f}}_{(\uparrow L)}(\theta) = X^{\text{f}}(L\theta)$$

Scrunches Spectrum

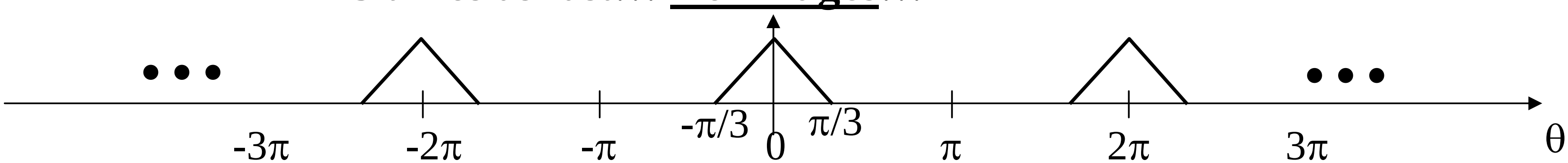
Example: DTFT for *L*-Fold Expansion

$L = 3$



Expansion Causes Images
to Appear in the [-π,π] Range

Here's what we'd have if we REALLY sampled 3 times as fast… **No Images**!!!



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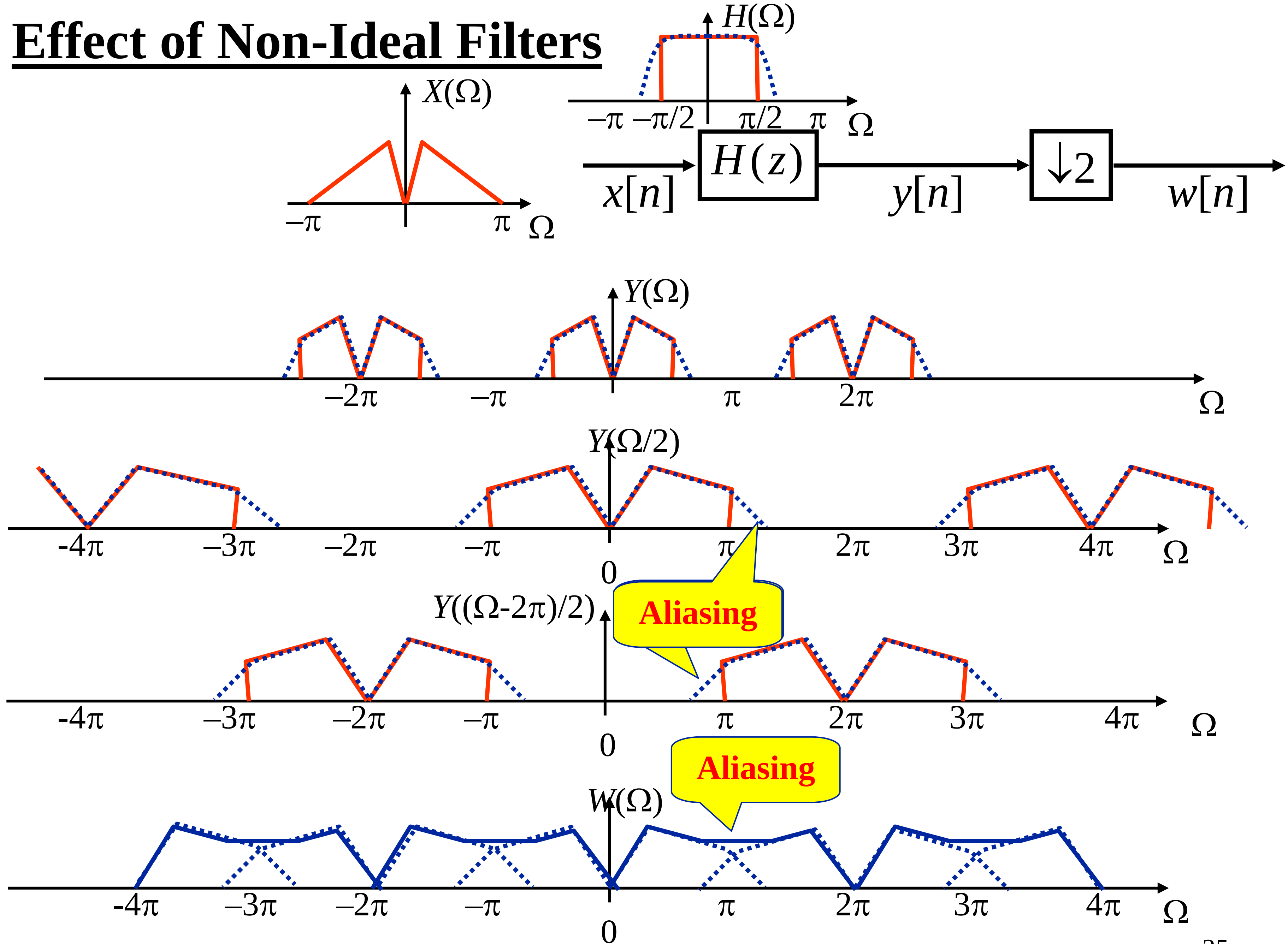
Effect of Non-Ideal Filters



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Summary

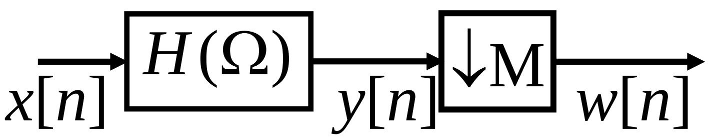
So… in practice we can change the rate of a signal… but there will always be some error due to non-ideal filters (both in the case of downsampling and in the case of upsampling).

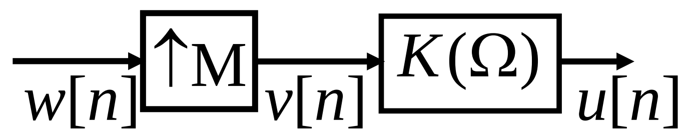
Generally, we can design the filters to make these errors negligible …

BUT… such filters are long FIR filters and that can lead to efficiency issues

Note: Similar analyses can be done for

- HPF followed by Decimation
- Interpolation followed by HPF





$$W(\Omega) = \frac{1}{M}\sum_{m=0}^{M-1} Y\left(\frac{\Omega - 2\pi m}{M}\right)$$

$$U(\Omega) = K(\Omega)W(M\Omega)$$

w/ $K(\Omega)$ having a passband width of π/M

$$Y(\Omega) = X(\Omega)H(\Omega)$$

w/ $H(\Omega)$ having a passband width of π/M

End