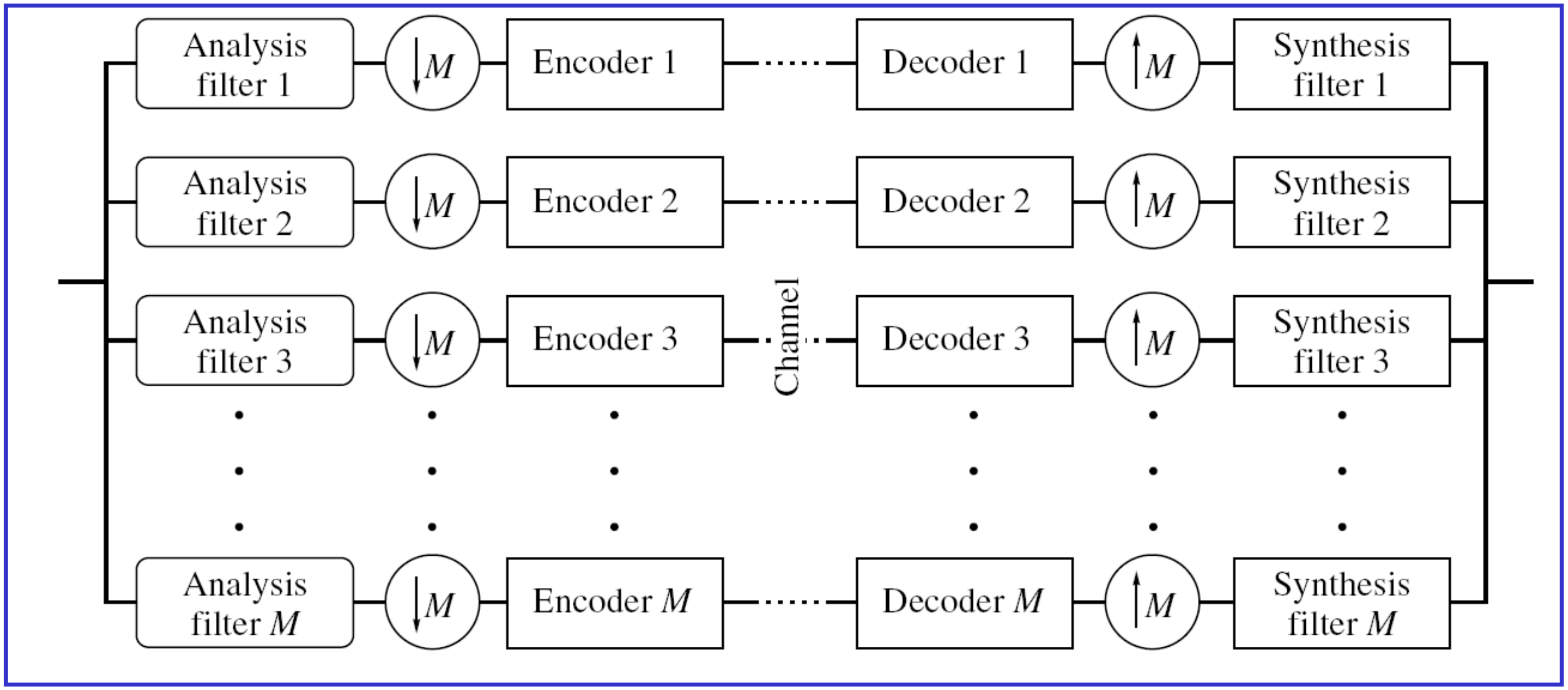
Ch. 14 Subband Coding

Perfect Reconstruction Filterbanks

Perfect Reconstruction for 2 Channels

Recall the general structure of subband coding:



To Design the Filters: Imagine removing the encoders/decoders… Then design so that the output is a "perfect reconstruction" of the input

"PR"

$$\hat{x}[n] = cx[n - n_o]$$

$$\hat{X}(z) = cX(z)z^{-n_o}$$

We'll limit here to $M = 2$ Channels…

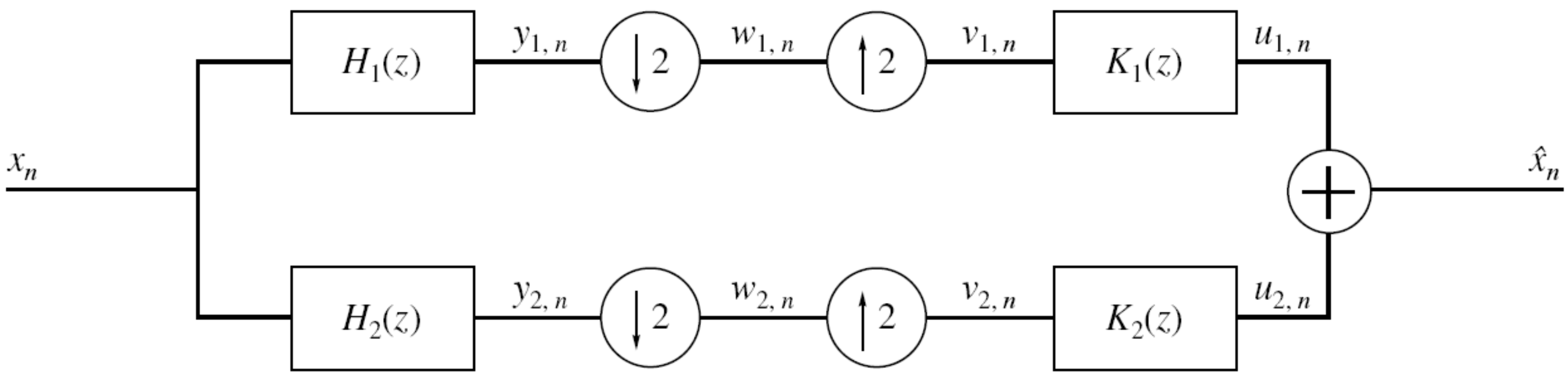
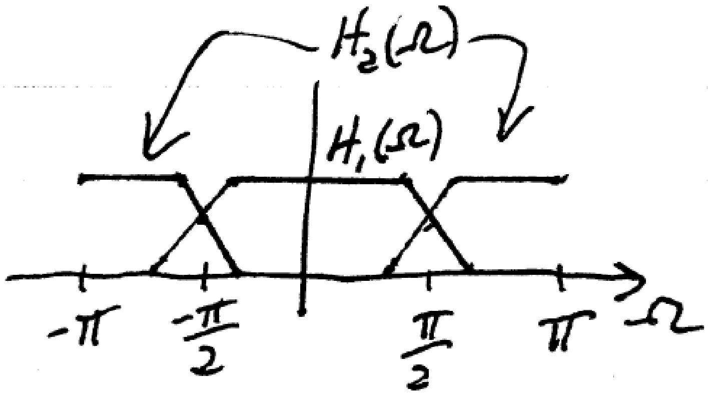


FIGURE 14. 17 **Two-channel subband decimation and interpolation.**

The "analysis" side filters are half-band LPF & HPF



<u>**Q**</u>: To ensure PR how do we choose:

<u>Analysis Filters</u>: $H_1(z)$, $H_2(z)$

<u>Synthesis Filters</u>: $K_1(z)$, $K_2(z)$

<u>**Note**</u>: If $H_1(z)$, $H_2(z)$, $K_1(z)$ & $K_2(z)$ are all <u>***ideal***</u> half-band filters then PR is easily achieved

But we can't build ideal filters… So is it even possible to <u>really</u> get PR????

Impact of Non-Ideal Filters

Stop-Band Issues

Analysis Filters: $H_1(z)$ & $H_2(z)$ will leave some content outside their half-band passbands that gets aliased into the passband after decimation.

Synthesis Filters: $K_1(z)$ & $K_2(z)$ will not completely eliminate the images created by upsampling that lie outside their half-band passbands.

Pass-Band Issues

Magnitude: For non-ideal filters the passbands are not perfectly flat and will change the shape of the signal's DTFT magnitude in the passband.

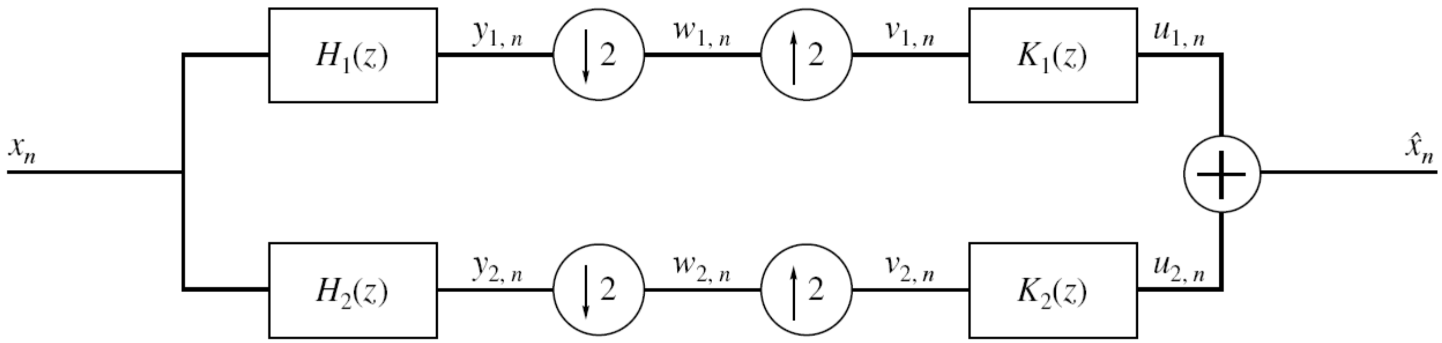
Phase: Because PR allows a delay and a delay corresponds to a linear phase response (as a function of frequency) it seems natural to focus on **linear phase filters** – which puts our focus on **FIR Filters**.

Our Goal: Choose filters such that the aliasing & imaging errors cancel out!!! (Fixes the stop-band issues)

Then… make what is left combine to give the desired composite passband to achieve the PR condition.

} Let's see how to do this mathematically

Math Analysis of PR Requirements



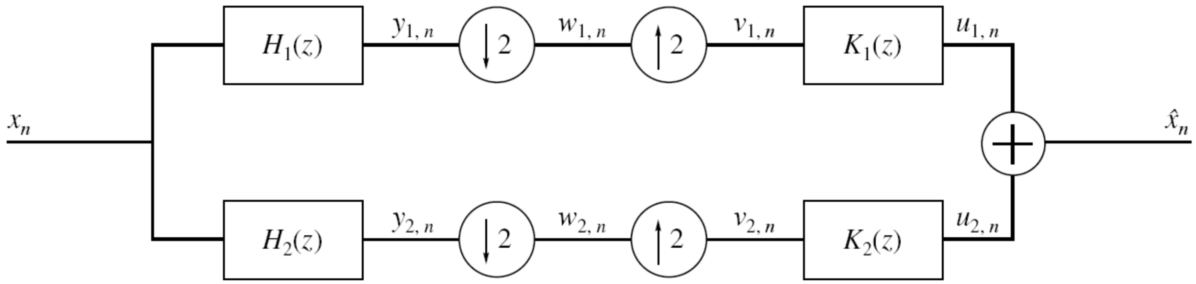
Start at input & work toward the output using z-transform methods:

Top Channel (Bottom Channel Similar):

$$Y_1(z) = H_i(z)X(z)$$ **Filter**

$$W_1(z) = \frac{1}{2}Y_1(z^{1/2}) + \frac{1}{2}Y_1(-z^{1/2})$$ **Down Sampling**

$$= \frac{1}{2}\left[H_1(z^{1/2})X(z^{1/2}) + H_1(-z^{1/2})X(-z^{1/2})\right]$$



$$V_1(z) = W_1(z^2)$$ **Up Sampling**

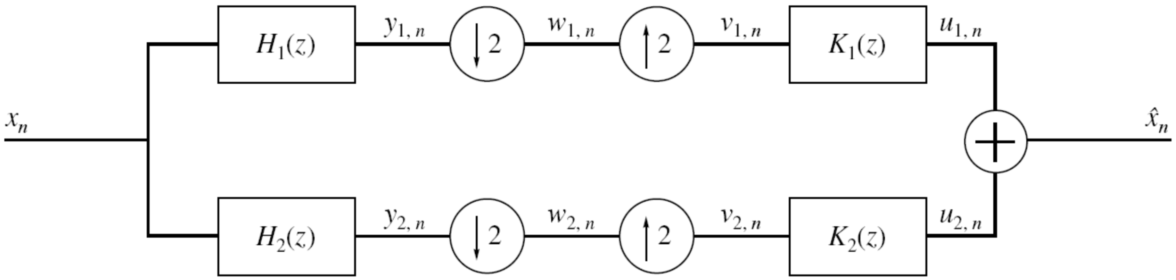
$$= \frac{1}{2}\left[H_1(z)X(z) + H_1(-z)X(-z)\right]$$

$$U_1(z) = K_1(z)V_1(z)$$ **Filter**

$$= \frac{1}{2}K_1(z)\left[H_1(z)X(z) + H_1(-z)X(-z)\right]$$

Now the output of the whole structure is:

$$\hat{X}(z) = U_1(z) + U_2(z)$$ **Summation**



Substitute results for $U_i(z)$ & Group $X(z)$ terms & group $X(-z)$ terms…

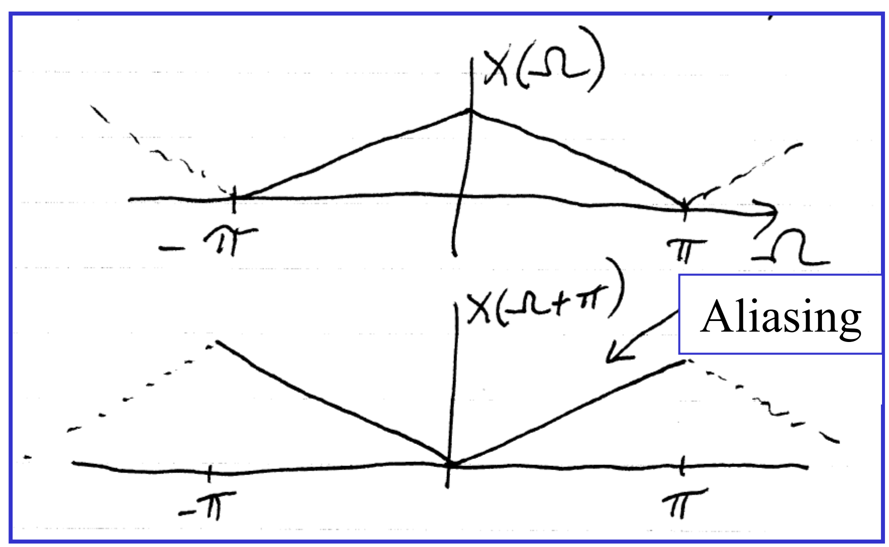
$$\hat{X}(z) = \underbrace{\frac{1}{2}\left[H_1(z)K_1(z) + H_2(z)K_2(z)\right]}_{\triangleq T(z)} X(z) + \underbrace{\frac{1}{2}\left[H_1(-z)K_1(z) + H_2(-z)K_2(z)\right]}_{\triangleq S(z)} X(-z)$$

$$\hat{X}(z) = T(z)X(z) + \underbrace{S(z)X(-z)}$$

Aliasing Term…
Don't Want It!

$$X(-z) = X(e^{j\pi}z) \xrightarrow{DTFT:\, z=e^{j\Omega}} X_{DTFT}(\Omega + \pi)$$

$$e^{j\pi}z\Big|_{z=e^{j\Omega}} = e^{j(\Omega+\pi)}$$



We want to eliminate this "Aliasing Term": ➔ $S(z) = 0$

$$H_1(-z)K_1(z) + H_2(-z)K_2(z) = 0$$

Can cancel aliasing term by choosing:

$$K_1(z) = H_2(-z) \quad \& \quad K_2(z) = -H_1(-z)$$

$$K_1(\Omega) = H_2(\Omega+\pi) \quad \& \quad K_2(\Omega) = -H_1(\Omega+\pi)$$

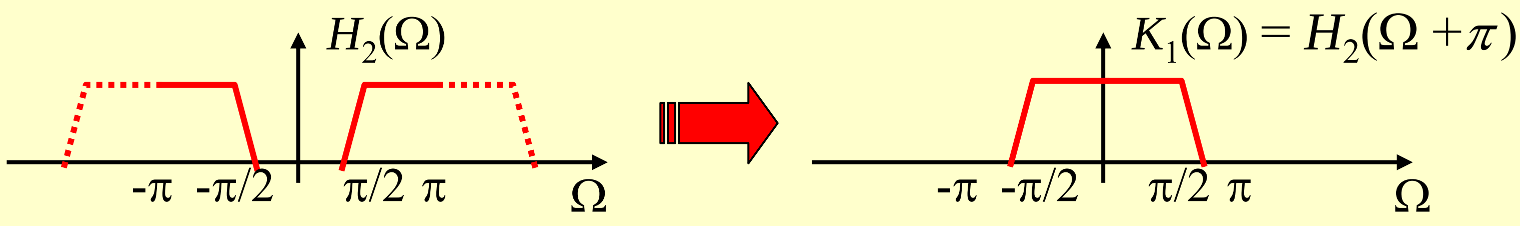
"ACC"
Aliasing
Cancellation
Condition

Doesn't constrain the filters… but constrains the relationship between the *K*'s and *H*'s

Note: Since…. $H_1(z)$ is lowpass & $H_2(z)$ is highpass…

we have: $K_1(z)$ is lowpass & $K_2(z)$ is highpass.

To see this: $K_1(z) = H_2(-z) \rightarrow K_1(z) = H_2(e^{j\pi}z) \rightarrow K_1(\Omega) = H_2(\Omega+\pi)$



Similarly: $K_2(z) = -H_1(-z) \rightarrow K_2(z) = -H_1(e^{j\pi}z) \rightarrow K_2(\Omega) = -H_2(\Omega+\pi)$

Once the *K*s are chosen this way we get

$$\hat{X}(z) = T(z)X(z)$$

$$= \frac{1}{2}\underbrace{\left[H_1(z)H_2(-z) - H_1(-z)H_2(z)\right]}_{\text{Want this } = Cz^{-n_o} \text{ for PR}} X(z)$$

So the condition the *H*s must meet for PR is:

$$\left[H_1(z)H_2(-z) - H_1(-z)H_2(z)\right] = Cz^{-n_o} \qquad (\star)$$

Note: the *K*s are chosen to cancel aliasing

the *H*s are chosen to give PR

Comment: For compression we not only want to cancel aliasing but we often need to minimize it in *each* channel… which requires all filters to have sharp transition bands and low stop bands

Why do we need this? Because in compression we often throw away some subbands (those having small energy)… and that upsets the balance used to cancel aliasing!

<u>Focus of Design Process</u>

So… our design process now focuses on designing the <u>analysis</u> filters $H_1(z)$ & $H_2(z)$ so that they meet (★) for PR

<u>Note</u>: The aliasing cancelation puts no constraint on the design of the filters… it only says: "if the analysis filters are *this*… then the synthesis filters must be *that*".

There are several design methods to get <u>analysis</u> filters $H_1(z)$ & $H_2(z)$ that give PR… various researchers have proposed these over the years.

We'll look at two:

- Quadrature Mirror Filters (QMF)
- Power Symmetric Filters
  - also called Conjugate Mirror Filters (CMF)

Quadrature Mirror Filters (QMF)

These were proposed in 1977 by Esteban & Galand

Their definition of QMF leads to:

- Useful filters for filterbanks
- But… not able to give PR (except in a trivial case)

QMF Definition: A pair of analysis filters are QMFs if

$$H_2(z) = H_1(-z)$$

$$-z\Big|_{z=e^{j\theta}} = e^{\pm j\pi}e^{j\theta} = e^{j\theta \pm \pi}$$

$$H_2(\theta) = H_1(\theta \pm \pi)$$

QMF Condition

Note: Once $H_1(z)$ is designed then the QMF condition nails down $H_2(z)$… …and remember that $K_1(z)$ & $K_2(z)$ are also nailed down by the ACC

So… enforcing QMF & ACC reduces the design problem to only designing $H_1(z)$

QMF "Facts"

1. If $H_1(z)$ is linear phase, so is $H_2(z)$

2. QMFs can only achieve PR if the $h_1[n]$ and $h_2[n]$ each have only 2 non-zero "taps"

   - E.g., $h_1[n] = [\ 1\ \ 1]$ or $h_1[n] = [\ 1\ \ 0\ \ 1]$ or $h_1[n] = [\ 1\ \ 0\ \ 0\ \ 1]$ etc.
   - <u>Note</u>: 2-tap Filters Stink! (See poor ½-band characteristics shown in Fig. 14.18)

3. If $H_1(z)$ has linear phase then $T(z)$… the analysis/synthesis total transfer function… also has linear phase.

   - Note that this is necessary for PR, where we need

Linear Phase

$$T(z) = Cz^{-n_o} \quad \Rightarrow \quad T(\Omega) = Ce^{-j\Omega n_o}$$

So… for QMF (w/ # taps > 2) we can't get the amplitude part of PR:

$$T(\Omega) = \underbrace{C(\Omega)}_{\substack{\text{real-valued} \\ \& \geq 0}} \; e^{-j\Omega n_o}$$

Linear Phase

Amplitude Distortion → **Not PR!**

QMF Design Process (One Way to Do It)

1. Once we get a design for the Analysis Filters… the $H_i(\Omega)$…

   Choose the Synthesis Filters … the $K_i(\Omega)$… to cancel aliasing

$$K_1(z) = H_2(-z) \quad \& \quad K_2(z) = -H_1(-z)$$

"ACC"

Not really part of QMF design… just what we do with them once we've designed them

$$\Rightarrow \quad T(z) = \frac{1}{2}\left[H_1(z)H_2(-z) - H_1(-z)H_2(z)\right]$$

2. Eliminate Phase Distortion by constraining the $H_i(\Omega)$ to be linear phase FIR filters… which ensures that you get:

$$T(\Omega) = C(\Omega)e^{-j\Omega n_o}$$

3. Enforce the QMF relationship…

$$H_2(z) = H_1(-z)$$

"QMF"

$$\Rightarrow \quad T(z) = \frac{1}{2}\left(\left[H_1(z)\right]^2 - \left[H_1(-z)\right]^2\right)$$

$T(z)$ now depends ***only*** on $H_1(z)$!!

4. Design $H_1(\Omega)$ to be a good LPF and to minimize the amplitude distortion of the ***end-to-end*** frequency response $T(\Omega)$. This can be done numerically by minimizing

$$J_\alpha(\mathbf{h}_1) = \alpha \int_{\Omega_s}^{\pi} |H_1(\Omega)|^2 \, d\Omega \quad + \quad (1-\alpha)\int_0^{\pi} \left[1 - |T(\Omega)|^2\right] d\Omega$$

See book "Numerical Recipes in C" for details on numerical minimization methods

Vector of Filter taps

Stop-Band "Energy"

Amplitude Distortion Measure

α controls relative priority of the two goals… $0 \le \alpha \le 1$

4. Once you have a $H_1(\Omega)$ that minimize J_α for your chosen α, then use it to generate all the other filters…

$$H_2(z) = H_1(-z)$$

"QMF"

$$K_1(z) = H_2(-z) = H_1(z)$$

$$K_2(z) = -H_1(-z)$$

"ACC"

Power Symmetric FIR Filters (or Conjugate Mirror Filters)

This ***Does* Allow** PR!!!

We'll get rid of aliasing the same way as before:

$$K_1(z) = H_2(-z) \quad \& \quad K_2(z) = -H_1(-z)$$ **"ACC"**

Error in Book in (14.75)

This gives $S(z) = 0$ (as before) and gives (as before)

$$T(z) = \frac{1}{2}\left[H_1(z)H_2(-z) - H_1(-z)H_2(z)\right]$$

Error in Book

Now… here is the new condition to use instead of QMF:

Only Use Odd *N*

$$H_2(z) = (-z)^{-N} H_1(-z^{-1})$$ **"CMF Z-D"**

where N = "Order" of the FIR filter $H_1(z)$

Recall: FIR has $h_1[n] = 0$ for $n \neq 0, 1, 2, \ldots, L\text{-}1$

Length = L Order = Length – 1 = L-1

Can show that "CMF" is equivalent to

$$h_2[n] = (-1)^n h_1(N-n)$$

"CMF T-D"

<u>For Order-N FIR with Odd N</u>…

$$\begin{array}{llllll} h_1[n]: & h_1[0] & h_1[1] & h_1[2] & \cdots & h_1[N] \\ h_2[n]: & h_1[N] & -h_1[N-1] & \cdots & h_1[1] & -h_1[0] \end{array}$$

Using "CMF Z-D" in $T(z)$ gives:

$$T(z) = \frac{1}{2} z^{-N} \left[\underbrace{H_1(z)H_1(z^{-1})}_{\triangleq R(z)} + H_1(-z)H_1(-z^{-1}) \right]$$

$$= \frac{1}{2} z^{-N} \left[\underbrace{R(z) + R(-z)}_{\text{Want = constant for PR}} \right]$$

Recall: 1. If $H_1(z) \leftrightarrow h_1[n]$ then $H_1(z^{-1}) \leftrightarrow h_1[-n]$

2. $F(z)G(z) \leftrightarrow f[n]*g[n]$

So… since $R(z) = H_1(z)H_1(z^{-1})$ ⇒ $R(z) \leftrightarrow \rho[n] = h_1[n] * h_1[-n]$

$$= \sum_{k=0}^{N} h_1[k]h_1[k+n]$$

"Time Auto Correlation" of $h_1[n]$

Note: $\rho[n] = 0$ for $|n| > N$

$\rho[-n] = \rho[n]$ (even symmetry)

So… (recalling that N is odd)…

$$R(z) = \rho[N]z^{N} + \rho[N-1]z^{N-1} + \cdots + \rho[1]z^{1} + \rho[0] + \rho[1]z^{-1} + \cdots + \rho[N]z^{-N}$$

$$R(-z) = -\rho[N]z^{N} + \rho[N-1]z^{N-1} - \cdots - \rho[1]z^{1} + \rho[0] - \rho[1]z^{-1} + \cdots - \rho[N]z^{-N}$$

Cancel **Cancel** **Cancel** **Cancel**

Odd-Indexed Terms Cancel when $R(z)$ & $R(-z)$ are added

$$R(z)+R(-z)= \text{ Only even-order terms}$$

$$=\rho[0]+\sum_{n=2,4,\ldots,N-1}\rho[n]\left(z^{n}+z^{-n}\right)$$

Want = constant for PR

$$\rho[n]=\begin{cases} C, & n=0 \\ 0, & n\ even, n\neq 0 \\ don't\ care, & n\ odd \end{cases}$$

Requirement for PR: $\rho[2n] = C\,\delta[n]$

$$\rho[2n]=\sum_{k=0}^{N}h_1[k]h_1[k+2n]=C\delta[n]$$

Time-Domain Requirement for PR

To convert this into Freq. Domain: $\underbrace{\mathcal{F}\{\rho[2n]\}}_{\text{DTFT of decimated sequence}} = \underbrace{\mathcal{F}\{C\delta[n]\}}_{=C}$ **(A)**

Dummy Variable

From F-D result for decimation: $\mathcal{F}\{\rho[2n]\} = \frac{1}{2}\left[R\left(\frac{\tilde{\Omega}}{2}\right) + R\left(\frac{\tilde{\Omega}-2\pi}{2}\right)\right]$

$\Omega \triangleq \frac{\tilde{\Omega}}{2}$

$= \frac{1}{2}\left[R(\Omega) + R(\Omega - \pi)\right]$ **(B)**

Now since $R(z) = H_1(z)\,H_1(z^{-1})$ The DTFT form is $R(\Omega) = H_1(\Omega)H_1(-\Omega)$

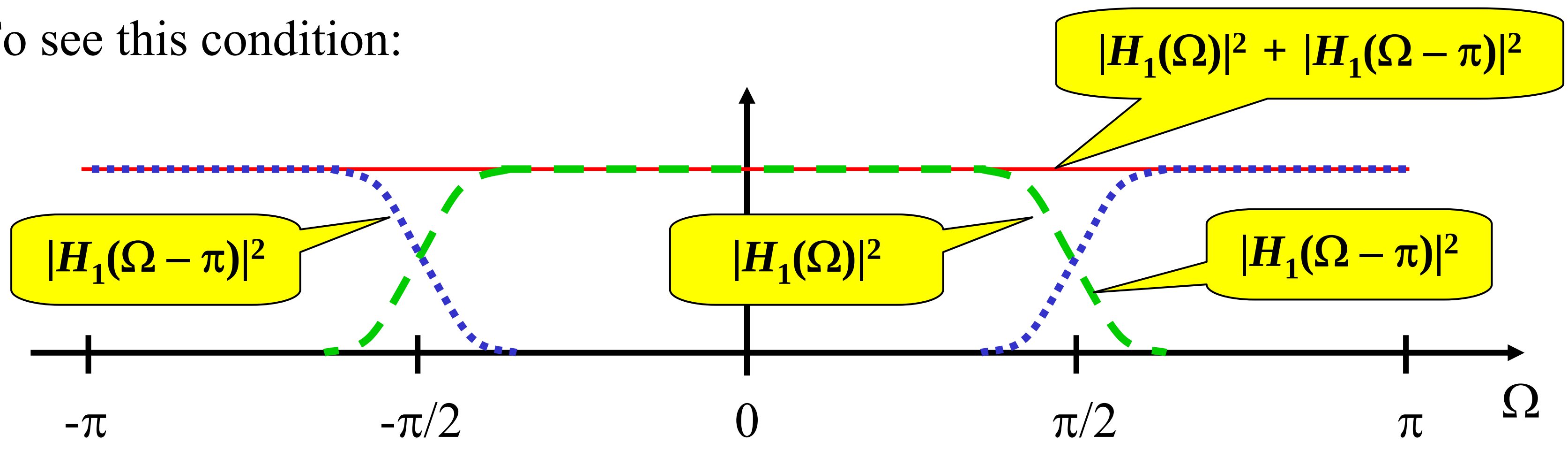
$= H_1(\Omega)H_1^*(\Omega)$

$R(\Omega) = |H_1(\Omega)|^2$ **(C)**

From (A) – (C) we get: $|H_1(\Omega)|^2 + |H_1(\Omega - \pi)|^2 = C$

Freq-Domain Requirement for PR

To see this condition:



Filters satisfying this are called…

"Power Symmetric Filters" or "Conjugate Mirror Filters"

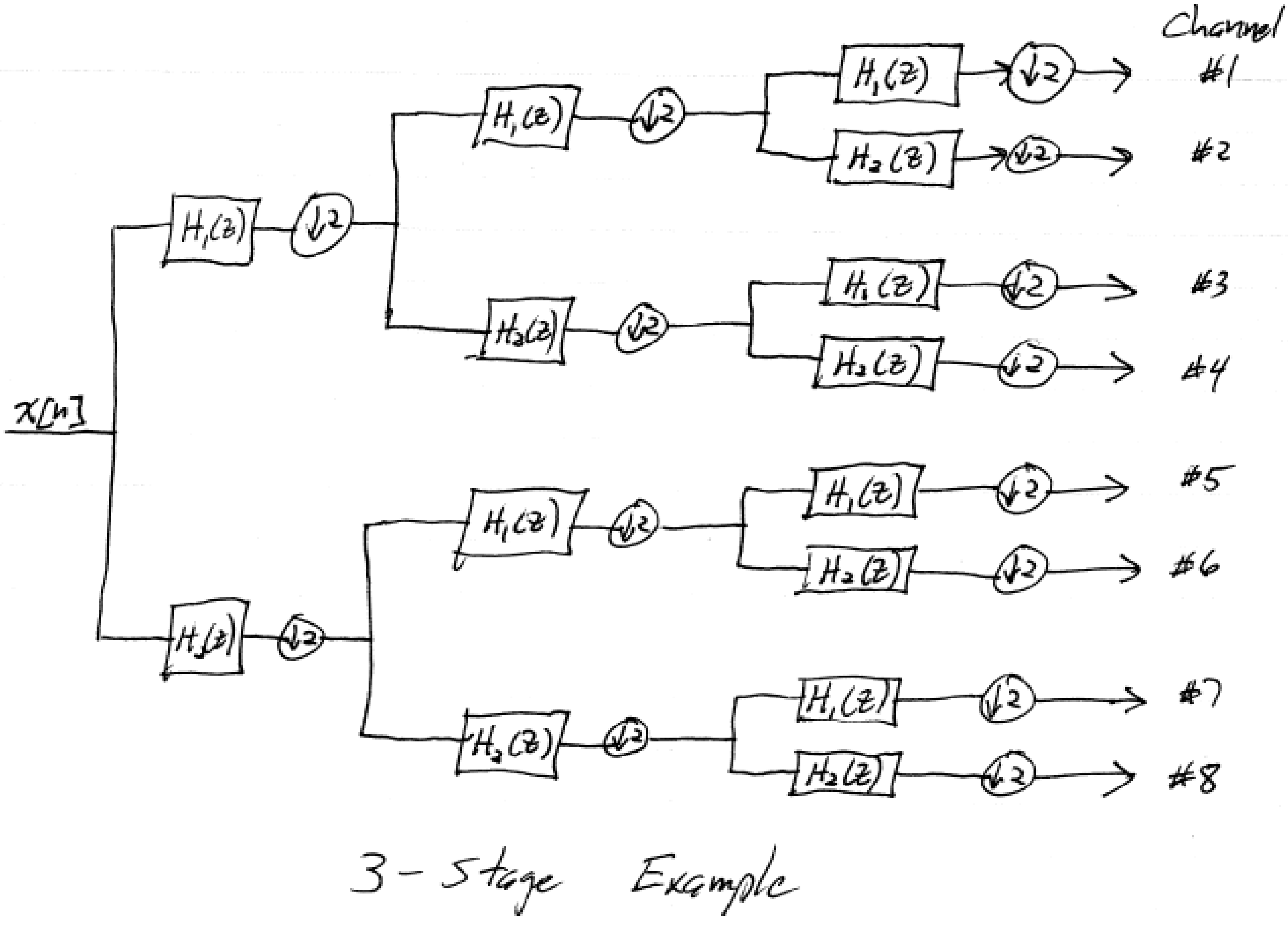
For Design Details… See Books on Filter Banks

Perfect Reconstruction for M Channels

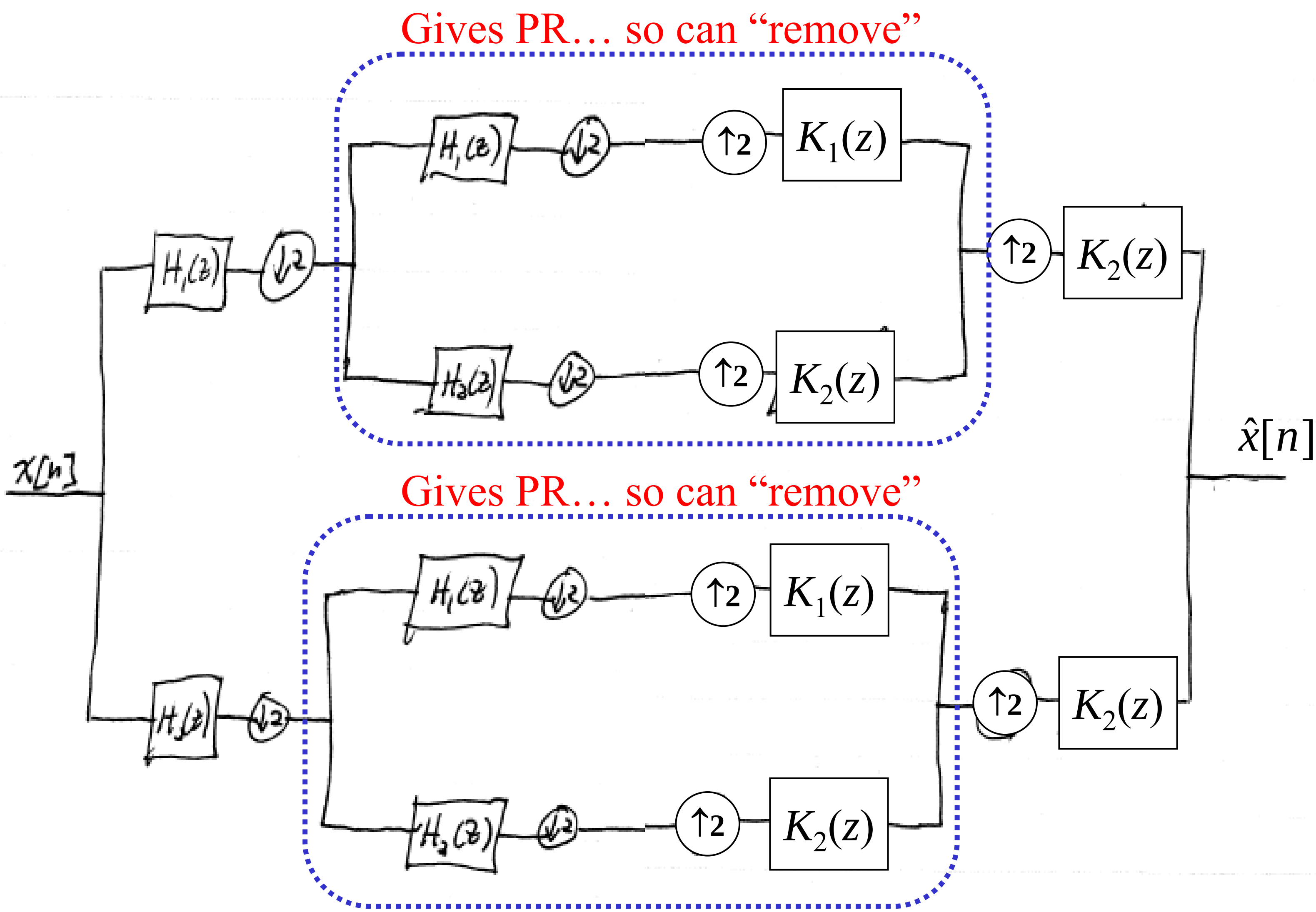
There are two ways to get PR for M > 2 Channels:

1. Extend all previous results to general M > 2 case
    - Same basic ideas but much more complicated
    - See books on Filter Banks
2. Cascade 2-Channel Stages…

Analysis Side of 3-Stage, 8-Channel PR Filter Bank

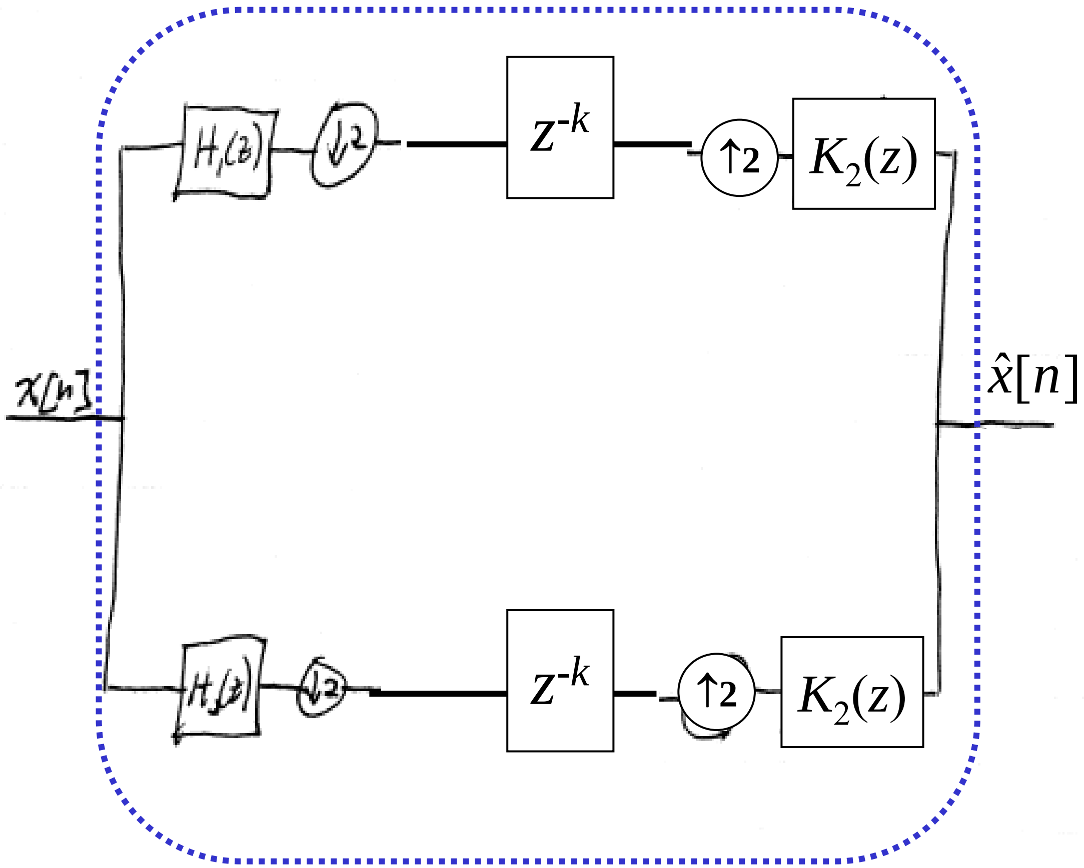


Design a 2-Channel PR Filterbank… Get M Channel PR:



After "removal" of Center:

Gives PR…



So… a cascade followed by the reverse cascade "collapses" to give M-Channel PR

What do the various channels in a cascaded analysis filter bank look like?

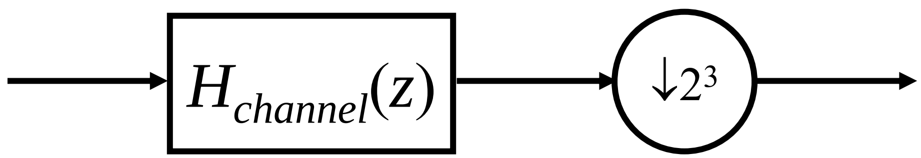
Can be shown that each channel has transfer function that looks like this (for the 3-stage case):

$$H_{channel}(z) = H_i(z) H_j(z^2) H_k(z^4)$$

1 or 2 (1st Stage)

1 or 2 (2nd Stage)

1 or 2 (3rd Stage)



The cascade method is useful but has a limitation:

If $H_i(z)$ has order N, then $H_i(z^2)$ has order $2N$, and $H_i(z^4)$ has order $4N$…

… and then the cascade of them has order $N + 2N + 4N$

Book has error… uses product here instead of add

BUT…. You only have N degrees of freedom in "choosing all those" $N + 2N + 4N$ coefficients!!!

Bit Allocation

Same ideas as for Bit Allocation for TC….

Each subband has its own quantizer and you want to allocate bits to the quantizers